AD 92567

Armed Services Technical Information Agency

Reproduced by

DOCUMENT SERVICE CENTER

KNOTT BUILDING, DAYTON, 2, OHIO

UNCLASSIFIED

NAVY DEPARTMENT REPORT

ON THE UNSTEADY MOTION OF A VISCOUS FLUID PAST A SEMI-INFINITE FLAT PLATE

BY

G. F. CARRIER & R. C. DiPRIMA

MARCH 1956

HARVARD UNIVERSITY
CAMBRIDGE, MASS.

CONTRACT NO. N5-ORI-07666

Navy Department Report

On the Unsteady Motion of a Viscous Fluid
Past a Semi-Infinite Flat Plate

by

G. F. Carrier and R. C. Di Prima

March 1956

Harvard University
Cambridge, Massachusetts
Contract No. N5-ORI-07666

On the Unsteady Motion of a Viscous Fluid Past a Semi-Infinite Flat Plate

G. F. Carrier and R. C. DiPrima

1. Introduction

The subject of how a boundary layer responds to fluctuations in the external flow about a steady mean has been discussed recently by several authors [6, 7, 8, 9, 10]. One reason for an interest in such problems arises in a study of the "Rijke tube" phenomenon [6, 7, 11]. There it appears to be important to know the phase lag in the heat transfer from a heated gauze situated in a stream with an unsteady velocity. Lighthill [7] investigated the oscillating flow problem using boundary layer theory and the Karman-Pohlhausen method. It is apparent that his results can not be valid near the leading edge of the object (say a heated ribbon). However most of the heat release takes place at the leading edge; hence it appears desirable to study the behaviour of the velocity fluctuations near the leading edge.

In this paper the viscous incompressible flow past a semi-infinite flat plate is investigated when the flow at infinity is $U_0 + U_1 \exp(i\omega t)$. The analysis is based on a system of equations derived by a modified Oseen linearization of the equations of motion [1, 2]. It is found that when $(\omega\nu)^{1/2} >> U_0$ and U_1 the flow field consists of the sum of a mean flow (due to U_0) and an oscillatory motion (due to $U_1 \exp(i\omega t)$). The phase advance in the oscillatory motion is $\pi/8$ at the leading edge and increases monotonically to $\pi/4$ in a distance approximately given by $(\omega/2\nu)^{1/2} x = 2.5$. The result for $(\omega/2\nu)^{1/2} x \geqslant 2.5$ is a shear wave type solution and corresponds to an infinite lamina performing harmonic oscillations in a viscous fluid.

In Sec. 5 the problem is considered when the interaction of the mean motion and the oscillatory motion can not be neglected. Expressions for the amplitude and phase advance of the unsteady skin friction near the leading edge are found. In particular the phase advance at the leading edge increases

monotonically from 0 to $\pi/8$ as the dimensionless parameter $(\omega\nu)^{1/2}/(c\,U_0)$ increases from 0 to ∞. Here c is a constant which is defined in Sec. 2. Far down the plate the solution approaches the shear wave solution.

2. Formulation of Problem

The equations governing a two-dimensional viscous, incompressible flow are

$$\left.\begin{aligned} u_x + v_y &= 0, \\ \rho(u_t + uu_x + vu_y) &= -p_x + \mu\Delta u, \\ \rho(v_t + uv_x + vv_y) &= -p_y + \mu\Delta v. \end{aligned}\right\} \tag{2.1}$$

Here x and y are the usual Cartesian coordinates, u and v are the corresponding velocities, p is the pressure, ρ the density (which is taken constant), $\nu = \mu/\rho$ is the kinematic viscosity, Δ is the Laplacian operator, and x, y, or t used as a subscript denotes partial differentiation with respect to that variable. We wish to determine the velocity field associated with a semi-infinite flat plate situated in an unsteady stream. For convenience we shall assume that the semi-infinite flat plate occupies the half line $y = 0$, $x \geqslant 0$. In particular we wish to consider this problem when the imposed velocity, U(t), far from the plate is of the form

$$U(t) = U_0 + U_1\, e^{i\omega t}, \tag{2.2}$$

where U_0 and U_1 are constants. The boundary conditions which u and v must satisfy are

$$\begin{aligned} &u = 0,\ v = 0 \ \text{ on } \ y = 0,\ x \geqslant 0, \\ &u \longrightarrow U_0 + U_1\, e^{i\omega t},\ v \longrightarrow 0 \end{aligned} \tag{2.3}$$

far from the plate.

In order to treat this problem we linearize Eqs. (2.1) by replacing the convective terms $uu_x + vu_y$, and $uu_x + vv_y$ by cUu_x and cUv_x. Here c is a constant which in general should depend upon the parameters involved in the problem. The case $U_1 = 0$, $c = 1$ gives the well known Oseen approximation. The modification of the Oseen method (i.e. $c \neq 1$) has been used in treating the steady flow over a semi-infinite plate [1]. Also it has been applied to several other steady flow problems, and its rationalization discussed in [2]. In [2] it was noted that for small Reynolds numbers that the value $c = .43$ gave results in good agreement with experiment in all cases considered. When the flow is unsteady, the choice of c is clearly more complicated. However in a recent publication [3] this type of linearization was used to compute the fluctuating heat release from a heated ribbon when $U(t)$ had the form given by Eq. (2.2), and the choice of $c = .43$ gave excellent results.

The form of $U(t)$ suggests that we write u and v as

$$\left.\begin{aligned} u(x, y, t) &= u_0(x, y) + u_1(x, y)\, e^{i\omega t}, \\ &= U_0\left\{1 + \chi_y(x, y)\right\} + U_1\left\{1 + \psi_y(x, y)\right\} e^{i\omega t}, \\ \text{and}\qquad & \\ v(x, y, t) &= v_0(x, y) + v_1(x, y)\, e^{i\omega t}, \\ &= - U_0\, \chi_x(x, y) - U_1\, \psi_x(x, y)\, e^{i\omega t}. \end{aligned}\right\} \qquad (2.4)$$

In order to obtain a solution of this form it is necessary to neglect quadratic terms in u_1; i.e. we must restrict ourselves to cases in which U_1 is small compared to either U_0 or $(\omega\nu)^{1/2}$. Then using the linearization discussed above and Eqs. (2.4) we obtain the following equations for χ and ψ,

$$\Delta\Delta\chi - a\Delta\chi_x = 0\ , \qquad (2.5)$$

$$\Delta\Delta\psi - a\Delta\psi_x - \beta^2\Delta\psi = a\Delta\chi_x\ . \qquad (2.6)$$

Here $a = cU_0/\nu$, and $\beta^2 = i\omega/\nu$. The boundary conditions that χ and ψ must satisfy are

$$\left.\begin{array}{l} \chi_x = 0, \ \psi_x = 0; \ \chi_y = -1, \ \psi_y = -1 \quad \text{on } y = 0, \ 0 \leq x \leq \infty, \\ \chi_x \rightarrow 0, \ \chi_y \rightarrow 0, \ \psi_x \rightarrow 0, \ \psi_y \rightarrow 0, \end{array}\right\} \quad (2.7)$$

far from the plate. The conditions ψ_x and $\chi_x = 0$ on $y = 0$, $0 \leq x \leq \infty$ clearly hold for all x and we satisfy them by requiring ψ and χ be constant on $y = 0$; in particular $\psi(x, o) = \chi(x, o) = 0$. As we shall see, when ω is very large the terms involving a in Eq. (2.6) may be neglected and we obtain homogeneous, independent equations for χ and ψ. That is to say, the velocity distribution is simply the flow due to a velocity U_0 at infinity plus that due to a velocity $U_1 e^{i\omega t}$ at infinity. In general of course these terms cannot be neglected, and hence the interaction of the mean flow and the time dependent flow must be considered. The term $a\Delta\psi_x$ represents the convective effect of the mean flow on the time dependent flow, while the non-homogeneous term $a\Delta\chi_x$ represents the convective effect of the time dependent flow on the mean flow.

To complete the formulation of our problem we must note that while χ, ψ, χ_y, ψ_y, χ_{yyy} and ψ_{yyy} are continuous across the plate χ_{yy} and ψ_{yy} which are proportional to the velocity gradient are discontinuous. We may express this discontinuity as

$$\chi_{yy}(x, o+) = -\chi_{yy}(x, o-) = \frac{1}{2} f_0(x) ,$$

$$\psi_{yy}(x, o+) = -\psi_{yy}(x, o-) = \frac{1}{2} f_1(x) .$$

The functions $f_0(x)$ and $f_1(x)$ are identically zero for $x < 0$. Now the skin friction, τ, is given by

$$\tau = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu \left\{\frac{\partial u_0}{\partial y} + \frac{\partial u_1}{\partial y} e^{i\omega t}\right\}_{y=0} .$$

Upon substituting for u_0 and u_1 from Eqs. (2.4) and using the above relations we may express τ in the form

$$\left.\begin{aligned} \tau &= \tau_0 + \tau_1 e^{i\omega t} \;, \\ &= \mu U_0 \frac{f_0(x)}{2} + \mu U_1 \frac{f_1(x)}{2} e^{i\omega t} \;. \end{aligned}\right\} \tag{2.8}$$

In what is to follow we shall be primarily concerned with the determination of $f_0(x)$ and $f_1(x)$.

3. Case $U_1 = 0$

In this case our problem reduces to a solution of the problem of the flow over a semi-infinite flat plate due to a flow U_0 at infinity. Thus we must solve Eq. (2.5) subject to the boundary conditions (2.7). Even when $U_1 \neq 0$ we still need the solution of this problem in order to obtain ψ (see Eq. (2.6)). The solution of this problem has been given in [1]. Since the analysis of the general problem will follow the lines of [1], but is much more complicated in its detail; and also since we need the results found in [1] it is convenient to use this simpler problem to illustrate the method of analysis that is to be used. Hence we shall repeat briefly part of the computations described in [1].

Our problem can be most conveniently treated by the use of Fourier transforms. We define

$$\overline{\chi}(\xi,\eta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\xi x + \eta y)} \chi(x,y)\,dx\,dy, \quad \overline{f}_0(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f_0(x)\,dx. \tag{3.1}$$

In all that is to follow a barred quantity will denote the Fourier transform of that quantity. Taking the Fourier transform of Eq. (2.5), and using the boundary conditions and discontinuity relation for χ we obtain

$$(\xi^2 + \eta^2)(\xi^2 + \eta^2 + ia\xi)\overline{\chi}(\xi,\eta) = i\eta \overline{f}_0(\xi) \;. \tag{3.2}$$

In order to avoid difficulties later it is desirable to write this equation as the limit as $k \longrightarrow 0$ of the equation[+]

$$\left\{\xi^2 + \eta^2 + k^2\right\}\left\{\eta^2 + (\xi + ia)(\xi - ik)\right\}\overline{\chi}(\xi,\eta) = i\eta \overline{f}_0(\xi). \qquad (3.3)$$

If we let $\overline{\chi}(\xi, y)$[#] denote the inverse transform of $\overline{\chi}(\xi,\eta)$ over η we obtain upon solving Eq. (3.3) and using simple contour integration that

$$\overline{\chi}(\xi, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\eta y}\,\overline{\chi}(\xi,\eta)\,d\eta = \int_{-\infty}^{\infty} e^{-i\xi x}\chi(x,y)\,dx\ , \qquad (3.4a)$$

$$= -\frac{y}{|y|}\,\frac{\exp\left\{-|y|\,(\xi^2+k^2)^{\frac{1}{2}}\right\} - \exp\left\{-|y|\,(\xi+ia)^{\frac{1}{2}}(\xi-ik)^{\frac{1}{2}}\right\}}{2i(a-k)(\xi-ik)}\,\overline{f}_0(\xi) \qquad (3.4b)$$

In particular we obtain from Eq. (3.4b) that

$$\overline{\chi}_y(\xi, y=0) = \frac{-\overline{f}_0(\xi)/2}{(\xi - ik)^{\frac{1}{2}}\left\{(\xi+ia)^{\frac{1}{2}} + (\xi+ik)^{\frac{1}{2}}\right\}}\ . \qquad (3.5)$$

A second equation for $\overline{\chi}_y(\xi, y=0)$ can be obtained by using our boundary conditions. First however the condition that $\chi_y = -1$ on the plate must be written as the limit, as $\alpha \longrightarrow 0$, of $\chi_y(x, o) = -\exp(-\alpha x)$ on $x \geqslant 0$. (The remarks made about k previously also apply to α.) Now differentiating (3.4a) with respect to y and setting $y = 0$ we have

$$\overline{\chi}_y(\xi, y=0) = \int_{-\infty}^{\infty} e^{-i\xi x}\chi_y(x, y=0)\,dx\ ,$$

$$= \overline{v}(\xi) - \frac{1}{i(\xi - i\alpha)}\ , \qquad (3.6)$$

where

$$\overline{v}(\xi) = \int_{-\infty}^{0} e^{-i\xi x}\chi_y(x, y=0)\,dx\ .$$

[+]Here and throughout the rest of this paper the symbol k will be used to represent a small, positive, real quantity which is to be shrunk to zero at an appropriate moment.

[#]We shall be careful to write $\overline{\chi}(\xi, y)$ if we mean the x transform of $\chi(x, y)$ and $\overline{\chi}(\xi,\eta)$ if we are referring to the two-dimensional transform.

The function $v(\xi)$ is of course not known. Equating (3.6) and (3.5) we obtain

$$\bar{v}(\xi) - \frac{1}{i(\xi - i\alpha)} = \frac{-\bar{f}_0(\xi)/2}{(\xi - ik)^{\frac{1}{2}}\left\{(\xi + ik)^{\frac{1}{2}} + (\xi + ia)^{\frac{1}{2}}\right\}} . \tag{3.7}$$

We can now determine $\bar{v}(\xi)$ and $\bar{f}_0(\xi)/2$ by the Wiener-Hopf technique. If we anticipate that $\chi_y(x, y = 0)$ decays exponentially in $(-x)$ on the negative real axis, then $\bar{v}(\xi)$ is an analytic function of the complex variable ξ in some upper half-plane (UHP) which includes the real axis. Similarly $\bar{f}_0(\xi)$ is analytic in some lower half-plane (LHP) including the real axis. Also $1/(\xi - i\alpha)$ is analytic in the LHP, $\mathrm{Im}(\xi) < \alpha$. And finally $K(\xi) = (\xi - ik)^{\frac{1}{2}}\left\{(\xi + ik)^{\frac{1}{2}} + (\xi + ia)^{\frac{1}{2}}\right\}$ is analytic in the strip $-k < \mathrm{Im}(\xi) < k$. Let us assume that we can split $K(\xi)$ into the product of two factors $K_+(\xi)$ and $K_-(\xi)$ such that $K_+(\xi)$ is analytic in the UHP, $\mathrm{Im}(\xi) > -k$; and $K_-(\xi)$ is analytic in the LHP $\mathrm{Im}(\xi) < k$. Then we may rewrite Eq. (3.7) as

$$\frac{\bar{f}_0(\xi)/2}{K_-(\xi)} - \frac{K_+(i\alpha)}{i(\xi - i\alpha)} = -\bar{v}(\xi)\,K_+(\xi) - \frac{K_+(i\alpha) - K_+(\xi)}{i(\xi - i\alpha)} = E(\xi) \tag{3.8}$$

The left-hand of this equation is analytic in the LHP, while the right side is analytic in an overlapping UHP, and hence they define an entire function $E(\xi)$. By using order conditions of the known functions at infinity and the fact that $E(\xi)$ is an entire function we can show that $E(\xi) \equiv 0$, hence

$$\frac{\bar{f}_0(\xi)}{2} = \frac{K_+(i\alpha)\,K_-(\xi)}{i(\xi - i\alpha)} \tag{3.9}$$

We recall once again that we are interested in evaluating $\bar{f}_0(\xi)/2$ in the limit $\alpha \longrightarrow 0$, $k \longrightarrow 0$.

In this particular problem $K_-(\xi)$ and $K_+(0)$ can be determined by inspection of $K(\xi)$ (we will not be so fortunate later on). We see that $K_-(\xi) = (\xi - ik)^{\frac{1}{2}}$ and $K_+(0) = (ia)^{\frac{1}{2}}$ in the limit as α and $k \longrightarrow 0$. Substituting

in (3.9) we may write $\bar{f}_0(\xi)$ as

$$\bar{f}_0(\xi)/2 = \frac{a^{\frac{1}{2}}}{(i\xi + k)^{\frac{1}{2}}} \tag{3.10}$$

We have written (3.10) in the preceding manner to indicate clearly that the singularity is in the UHP. In the limit as $k \longrightarrow 0$ we find by the conventional inversion formula that

$$f_0(x)/2 = \begin{cases} \left(\frac{a}{\pi x}\right)^{\frac{1}{2}} = \left(\frac{c\,U_0}{\pi \nu x}\right)^{\frac{1}{2}} & x > 0 \;, \\ 0 & x < 0 \;. \end{cases} \tag{3.11}$$

As mentioned previously the choice of $c = .43$ gives good agreement with the non-linear theory and with experimental results for small Reynolds number. However in this case the Reynolds number is infinite and the result which is consistent with the Blasius solution is given by $c = .35$ [1]. It is clear that the numerical choice of c can not affect the character of the result, and hence for any value of c in this range the result will describe the basic features of the flow. Finally knowing $\bar{f}_0(\xi)$ it is possible to invert $\bar{\chi}(\xi, y)$ and obtain

$$\chi(x, y) = \frac{U_0}{\nu}\left\{\frac{2(r+x)^{\frac{1}{2}}}{(2\pi a)^{\frac{1}{2}}}\left[1 - e^{-a(x-r)/2}\right] + y \operatorname{erfc}\left[\frac{a(r-x)}{2}\right]^{\frac{1}{2}}\right\}^{+},$$

where $r^2 = x^2 + y^2$.

From the preceding results and the definition of τ_0, Eq. (2.8), we have

$$\tau_0 = \mu U_0 \left(\frac{c\,U_0}{\pi \nu x}\right)^{\frac{1}{2}} \tag{3.12}$$

We can use Eq. (3.12) to determine the skin friction when $U_1 \neq 0$, by expanding τ_0 in a Taylor series about U_0. We have, if $\tau(U_0 + U_1)$ denotes the skin friction

---

+This result can be obtained from Eq. (21) of [1]. It also was given explicitly by Lagestrom, Cole and Trilling [4].

for a flow $U_0 + U_1$ at infinity that

$$\tau(U_0 + U_1) = \tau_0 + U_1 \left.\frac{\partial \tau_0}{\partial U_0}\right|_{U_0} + \ldots$$

$$= \left(1 + \frac{3}{2}\frac{U_1}{U_0}\right)\tau_0 + \ldots$$

This result is correct up to quadrate terms in U_1, and since this is the same as the accuracy in Eq. (2.6) we may use this result to compute τ_1 when $\omega = 0$. We have

$$\tau_1(\omega = 0) = \frac{3}{2}\mu U_1 \left(\frac{c\,U_0}{\pi \nu x}\right)^{\frac{1}{2}}, \quad \frac{1}{2}\, f_1(x, \omega = 0) = \frac{3}{2}\left(\frac{c\,U_0}{\pi \nu x}\right)^{\frac{1}{2}}. \qquad (3.13)$$

These results will be of interest in the discussion in Sec. 5.

4. <u>Case $U_0 = 0$</u>

In this section we require that $U_1 \ll (\omega\nu)^{\frac{1}{2}}$ as indicated just after Eqs. (2.4). When $U_0 = 0$, Eq. (2.6) reduces to

$$\Delta\Delta\,\psi - \beta^2\,\Delta\psi = 0 \;. \qquad (4.1)$$

We should note that this equation gives the first approximation to the time dependent flow even when $U_0 \neq 0$ if the parameter $\sigma = (\omega\nu)^{\frac{1}{2}}/c\,U_0$ is large. This can be seen by introducing appropriate dimensionless variables in Eqs. (2.5) and (2.6) and then expanding ψ in powers of σ. The length scale that one should use is $(\nu/\omega)^{\frac{1}{2}}$.

To solve (4.1) we proceed in exactly the same manner as described in Sec. 3. The Fourier transform of Eq. (4.1) is

$$(\xi^2 + \eta^2 + \beta^2)(\xi^2 + \eta^2 + k^2)\,\bar{\psi}(\xi, \eta) = i\eta\,\bar{f}_{10}(\xi)\,. \qquad (4.2)$$

We have written $\bar{f}_{10}(\xi)$ instead of $\bar{f}_1(\xi)$ to indicate that this is the special case $U_0 = 0$, (i.e. the limit case $\sigma \to \infty$). From Eq. (4.2) we obtain

$$\bar{\psi}(\xi, y) = -\frac{y}{|y|}\,\frac{\exp\left\{-|y|(\xi^2 + k^2)^{\frac{1}{2}}\right\} - \exp\left\{-|y|(\xi^2 + \beta^2)^{\frac{1}{2}}\right\}}{(\beta^2 - k^2)}\,\frac{\bar{f}_{10}(\xi)}{2}\,, \qquad (4.3)$$

and finally

$$\frac{\bar{f}_{10}(\xi)}{2} = \frac{N_+(i\alpha)\, N_-(\xi)}{i(\xi - i\alpha)} \quad , \tag{4.4}$$

where $N(\xi) = (\xi^2 + \beta^2)^{\frac{1}{2}} + (\xi^2 + k^2)^{\frac{1}{2}}$ is analytic in the strip $-k < \mathrm{Im}(\xi) < k$.

Before we proceed we must determine $N_-(\xi)$. In the previous section we could determine $K_-(\xi)$ by inspection; here the situation is more difficult. The details of the computation of $N_-(\xi)$ are presented in Appendix A. It is found that

$$N_-(\xi) = \exp\left\{- \frac{}{2\pi i} \int_0^{\xi} F(\xi)\, d\xi\right\}, \tag{4.5}$$

where in the limit as $k \longrightarrow 0$

$$F(\xi) = \frac{1}{(\xi^2 + \beta^2)^{\frac{1}{2}}} \ln \frac{\beta + (\xi^2 + \beta^2)^{\frac{1}{2}}}{\beta - (\xi^2 + \beta^2)^{\frac{1}{2}}} \quad . \tag{4.6}$$

The behaviour of $F(\xi)$ in the complex plane is quite complicated. The points $\xi = i\beta$ and $\xi = 0$ are branch points of the function; however the behaviour of the logarithmic term in (4.6) must be such that $\xi = -i\beta$ is not a branch point. (Recall $N_-(\xi)$ must be analytic in the LHP.) Thus we choose the branch of the logarithm by defining the logarithmic term in (4.6) to be zero at $\xi = -i\beta$, in which case it becomes $-2\pi i$ at $\xi = +i\beta$. This can be seen by writing $F(\xi) = 2(\xi^2 + \beta^2)^{\frac{1}{2}} \ln \left\{[\beta + (\xi^2 + \beta^2)^{\frac{1}{2}}]/e^{\pi i/2}\xi\right\}$ where we have taken $(-1) = e^{\pi i}$. As our branch cut we shall take the entire positive imaginary axis. The determination of ξ and $(\xi^2 + \beta^2)^{\frac{1}{2}}$ on the branch cuts is given in Appendix B where the results are used in evaluating $f_{10}(x)$. Finally from Eq. (4.5) it is clear that $N_-(o) = 1$, and since when $k \longrightarrow 0$, $N(o) = \beta$ we see that $N_+(o) = \beta$.

We may now compute $f_{10}(x)$. Upon substituting in Eq. (4.4) and using the usual inversion formula we have, letting $k \longrightarrow 0$, $\alpha \longrightarrow 0$ that

$$\frac{1}{2}\, f_{10}(x) = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{i\xi} \exp\left\{- \frac{1}{2\pi i} \int_0^{\xi} F(\xi)\, d\xi\right\} , \tag{4.7}$$

where the path of integration must pass below the origin. This integral is evaluated in Appendix B. We obtain

$$\frac{1}{2} f_{10}(x) = \frac{\beta}{\pi} \left\{ \pi + \int_0^1 e^{-yx'} H(y)\, dy + \int_1^\infty e^{-yx'} \frac{[y + (y^2-1)^{\frac{1}{2}}]^{\frac{1}{2}}}{1} P(y)\, dy \right\}. \quad (4.8)$$

Here $x' = \beta x = (1 + i)(\omega/2\nu)^{\frac{1}{2}} x$,

$$H(y) = \exp\left\{- \frac{1}{\pi}\left[t \ln \cot \frac{t}{2} + \int_0^t \frac{\theta}{\sin\theta}\, d\theta\right]\right\}, \qquad t = \arc\sin y \quad ,$$

$$P(y) = \exp\left\{ - \frac{1}{\pi} \int_0^m \frac{\theta}{\sin\theta}\, d\theta \right\} \qquad , \quad m = \arc\sin (1/y) \quad ,$$

and $0 < y < 1$ implies $0 < t < \pi/2$; also $1 < y < \infty$ implies $\pi/2 > m > 0$.

It is interesting to note, using Eq. (4.8) that $f_{10}(x)$ is of the form β times a function of βx. The expression for $f_{10}(x)$ given by Eq. (4.8) may be used for numerical computation when $(\omega/2\nu)^{\frac{1}{2}} x$ is neither large nor small. Of course the second integral must be converted into an integration over a finite range but this is not a difficult matter. Using Eq. (4.8), $f_{10}(x)/2$ has been computed for $hx = 1/2$ and 1, where $h = (\omega/2\nu)^{\frac{1}{2}}$. The results of this computation are given at the end of this section. When $x = 0$ it is clear that the second integral in (4.8) does not converge; this of course is not surprising since we could anticipate from the result of Sec. 3 that $f_{10}(x)$ should behave like $x^{-\frac{1}{2}}$ near the leading edge of the plate.

To complete our investigation of this case we shall obtain expressions for the behaviour of $f_{10}(x)$ for large and small values of hx. To do this we first note, since $f_{10}(x)$ is identically zero for $x < 0$, that the Fourier transform of $f_{10}(x)$ is essentially the usual Laplace transform but rotated through 90°. Hence we may make use of the Tauberian theorems relating the behaviour of the Laplace inverse at the origin to the behaviour of the function at infinity, and the behaviour of the Laplace inverse at infinity to the behaviour of the function at the origin. (See [5].) Thus we must study the behaviour of $N_-(\xi)$

for large ξ and small ξ.

The expression for $N_-(\xi)$ given by Eq. (4.5) can be written in the more convenient form

$$N_-(\xi) = \left\{s + (1+s^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}} \; e^{\pi i/4} \; \exp\left\{- \frac{i}{\pi}\int_{\theta}^{\infty} \ell n(\coth \theta/2)\, d\theta\right\}, \quad (4.9)$$

where $s = \xi/\beta$ and $\theta = \sinh^{-1} s$. This result is obtained by rationalizing $\{\beta + (\xi^2+\beta^2)^{\frac{1}{2}}\}/\{\beta - (\xi^2+\beta^2)^{\frac{1}{2}}\}$ in (4.6), and then making the substitution $s = \sinh\theta$. It is not difficult to show from (4.9) that as $s \longrightarrow 0$

$$N_-(\xi) \longrightarrow 1 \, , \quad (4.10a)$$

and that as $s \longrightarrow \infty$

$$N_-(\xi) = \left\{\left(\frac{2\xi}{\beta}\right)^{\frac{1}{2}} - \frac{e^{\pi i/2}}{\pi}\left(\frac{2\beta}{\xi}\right)^{\frac{1}{2}}\right\} e^{\pi i/4} + 0[(\beta/\xi)^{\frac{3}{2}}] \, . \quad (4.10b)$$

Using these results and Eq. (4.4) we obtain that as $x \longrightarrow \infty$

$$\frac{1}{2}\, f_{10}(x) \longrightarrow \beta = \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} e^{\pi i/4} \quad , \quad (4.11a)$$

and as $x \longrightarrow 0$

$$\frac{1}{2}\, f_{10}(x) = \left(\frac{2\beta}{\pi x}\right)^{\frac{1}{2}} \left\{1 + \frac{2\beta x}{\pi} + 0[(\beta x)^2]\right\} . \quad (4.11b)$$

In particular for very small βx

$$\frac{1}{2}\, f_{10}(x) \longrightarrow \left\{\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} \frac{2}{\pi x}\right\}^{\frac{1}{2}} e^{\pi i/8} \quad .$$

As could have been anticipated Eq. (4.11a) is the result for an infinite lamina undergoing simple harmonic motion. (See [6].) This result represents a wave of transversal vibration, i.e. a shear wave. Exactly how far down the plate it is necessary to go before (4.11a) is valid can be obtained by evaluating $f_{10}(x)$ from Eq. (4.8). The argument and a dimensionless modulus of $f_{10}(x)$ have been computed using Eqs. (4.8) and (4.11b). The results are tabulated in the table at the end of this section, and are exhibited graphically in Figs. 3 and 4. It appears from Fig. 3 that Eq. (4.11a) is an excellent representation for $f_{10}(x)$ for $hx > 2.5$ approximately. Also from the tabulated results it appears that Eq. (4.11b) is valid for $hx \leq 1/2$ approximately.

To summarize our results we have, under the assumptions $(\omega\nu)^{\frac{1}{2}} >> U_0$ and $(\omega\nu)^{\frac{1}{2}} >> U_1$, that the skin friction τ is

$$\tau = \mu U_0 \frac{f_0(x)}{2} + \mu U_1 \left\{ f_{10}(x) + 0\left(\frac{1}{\sigma}\right)\right\} e^{i\omega t} \tag{4.12}$$

where $f_0(x)/2$ is given by Eq. (3.11) and $f_{10}(x)/2$ is given by Eq. (4.8). Near the leading edge of the plate, τ takes the form

$$\tau = \mu U_0 \left(\frac{c\,U_0}{\pi\nu x}\right)^{\frac{1}{2}} + \mu U_1 \left\{\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}\left(\frac{2}{\pi x}\right)\right\}^{\frac{1}{2}} e^{i(\omega t + \pi/8)} \quad . \tag{4.13}$$

Far down the plate, i.e. $hx > 2.5$ approximately,

$$\tau = \mu U_0 \left(\frac{c\,U_0}{\pi\nu x}\right)^{\frac{1}{2}} + \mu U_1 \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} e^{i(\omega t + \pi/4)} \quad . \tag{4.14}$$

Thus the flow consists of the sum of a mean flow plus an oscillatory motion which is independent of the mean motion. The phase advance in the skin friction increases monotonically from $\pi/8$ at the leading edge to $\pi/4$ in a distance of $hx = 2.5$ approximately.

TABLE

$(\omega/2\nu)^{\frac{1}{2}} x$	$\mathrm{Arg}\{f_{10}(x)\}$	$\lvert f_{10}(x)\rvert/2(\omega/2\nu)^{\frac{1}{2}}$	Evaluated by
$\to 0$	22.5°	$\to \infty$	
.1	25.93°	3.19	Asymptotic Formula (Eq. (4.11b)
.25	29.97°	2.20	" " "
.50	36°	1.77	" " "
.50	34.65°	1.80	Numerical Computation Eq. (4.8)
1.0	39.15°	1.58	" " "
$\to \infty$	45°	$2^{\frac{1}{2}}$	

5. Case $U_0 + U_1\ e^{i\omega t}$

Now we are concerned with solving Eq. (2.6) without any restriction on the parameters involved. In theory we could, of course, proceed exactly as we have in the past two sections. However because of the non-homogeneous term as well as the more complicated differential operator which is applied to ψ we should expect the details of the computation to be more difficult. In actual fact a direct application of the technique used in the previous sections does present some rather formidable problems; hence it appears desirable to attack this problem in a slightly different manner.

First we note that ψ can be expressed as the sum of an appropriate solution of the homogeneous equation plus a solution of the non-homogeneous equation. Hence we can write

$$\psi(x, y) = \psi_1(x, y) + \psi_2(x, y) \quad , \tag{5.1}$$

where ψ_1 is a solution of the homogeneous equation which also satisfies the boundary condition $\partial\psi_1/\partial y = -1$ on the plate. Then the solution, ψ_2, of the non-homogeneous equation (2.6), must satisfy the condition $\partial\psi_2/\partial y = 0$ on the plate. The discontinuity condition in ψ_{yy} can be expressed as $f_1(x) = f_{11}(x) + f_{12}(x)$ where

$$\psi_{1yy}(x, 0+) = -\psi_{1yy}(x, 0-) = \frac{1}{2} f_{11}(x) \quad ,$$

$$\psi_{2yy}(x, 0+) = -\psi_{2yy}(x, 0-) = \frac{1}{2} f_{12}(x) \quad .$$

First we shall treat ψ_1. In determining $f_{11}(x)$ we shall proceed exactly as before. The Fourier transform of ψ_1 is

$$(\eta^2 + \xi^2 + k^2)\left\{\eta^2 + \xi^2 + ia\xi + \beta^2\right\} \bar{\psi}_1(\xi, \eta) = i\eta \bar{f}_{11}(\xi) \, , \tag{5.2}$$

Solving for $\psi_1(\xi,\eta)$ and inverting over η we obtain

$$\psi_1(\xi, y) = -\frac{y}{|y|}\ \frac{\exp\left\{-|y|(\xi^2+k^2)\right\} - \exp\left\{-|y|(\xi^2+ia\xi+\beta^2)^{\frac{1}{2}}\right\}}{(ia\xi+\beta^2-k^2)}\ \frac{\bar{f}_{11}(\xi)}{2}\ . \tag{5.3}$$

Then

$$\frac{\bar{f}_{11}(\xi)}{2} = \frac{M_+(i\alpha)\,M_-(\xi)}{i(\xi - i\alpha)}\ , \tag{5.4}$$

where $M(\xi) = (\xi^2 + ia\xi + \beta^2)^{\frac{1}{2}} + (\xi^2 + k^2)^{\frac{1}{2}}$. The splitting of $M(\xi)$ can be carried out in the same manner as was done in Sec. 4, hence

$$\ell n\, M_-(\xi) = \frac{1}{2\pi i}\int_{\Gamma_2} \frac{\ell n\left\{(z^2+iaz+\beta^2)^{\frac{1}{2}} + (z^2+k^2)^{\frac{1}{2}}\right\}}{z-\xi}\ dz\ . \tag{5.5}$$

If we multiply the top and bottom of Eq. (5.4) by $N_+(o)\,N_-(\xi)$, note that $M_+(o) = M(o)/M_-(o) = \beta M_-(o)$, $N_+(o) = \beta$, and $N_-(o) = 1$, and use Eq. (4.4) we may express $\bar{f}_{11}(\xi)$ as

$$\frac{\bar{f}_{11}(\xi)}{2} = \left\{\frac{M_-(\xi)/N_-(\xi)}{M_-(o)/N_-(o)}\right\}\ \frac{\bar{f}_{10}(\xi)}{2}\ . \tag{5.6}$$

Thus the transform of the skin friction contribution from the homogeneous solution can be expressed as a function of ξ times the transform of the skin friction when $U_O = 0$. Equation (5.6) will be examined in more detail shortly, but now let us turn our attention to determining the non-homogeneous solution of Eq. (2.6).

Using Eq. (2.5) it is clear that $-(a/\beta^2)\,\chi_x$ is a solution of Eq. (2.6). However an inspection of the properties of χ as given in Sec. 3 shows that for $y = 0$, $\psi_2 = -(a/\beta^2)\,\chi_x$ will lead to a velocity which becomes infinite at the leading edge of the plate, i.e. $\chi_{xy}(x, o)$ behaves like $x^{-\frac{1}{2}}$ for small $x > 0$. This, of course, we can not accept. But we can add solutions of the homogeneous equation to this solution, and the result will still be a solution of the non-homogeneous

equation. Hence we might be able to construct an appropriate non-homogeneous solution by finding a homogeneous solution which will destroy the undesirable singularity in χ_x. Thus we take ψ_2 to be of the form

$$\psi_2(x, y) = -\frac{a}{\beta^2}\left\{\chi_x(x, y) + \phi(x, y)\right\} , \qquad (5.7)$$

where $\phi(x, y)$ must satisfy the homogeneous equation $(\Delta - a\partial/\partial x - \beta^2)\Delta\phi = 0$ and also remove the singularity in χ_{xy}. To describe our problem completely let us specify exactly the conditions which we must require of ψ_2. The first condition is $\psi_2(x, o) = 0$. Secondly we require that $\partial\psi_2/\partial y = 0$ on $y = 0$, $x > 0$, since $\partial\psi_1/\partial y = -1$ there. Finally $\partial^2\psi_2/\partial y^2 = 0$ on $y = 0$, $x < 0$, since $f_1(x) \equiv 0$ for $x < 0$. And of course also ψ_2's behaviour is such that its contribution to the velocity is finite at all points. The first three of these requirements are clearly satisfied by χ_x but not the last as mentioned earlier. Hence ϕ must satisfy the first three requirements and also fix the singularity in χ_x. The determination of $\phi(x, y)$ and in particular $\bar{\phi}_{yy}(\xi, o)$ is given in Appendix C.

Now the transform of $f_{12}(x)/2$ is given by

$$\frac{\bar{f}_{12}(\xi)}{2} = -\frac{a}{\beta^2}\left\{\bar{\chi}_{xyy}(\xi, y = 0) + \phi_{yy}(\xi, y = 0)\right\} , \qquad (5.8)$$

and noting that $\bar{\chi}_{xyy}(\xi, y = 0) = i\xi\,\bar{f}_0(\xi)/2$, and substituting for $\bar{\phi}_{yy}(\xi, y = 0)$ from Appendix C we have

$$\frac{\bar{f}_{12}(\xi)}{2} = -\frac{a}{\beta^2}\left\{i\xi\,\frac{\bar{f}_0(\xi)}{2} - e^{\pi i/4}\,a^{\frac{1}{2}}\,(\xi - ia_1)^{\frac{1}{2}}\,Q_-(\xi)\right\} , \qquad (5.9)$$

where $a_1 = \left\{(a^2 + 4\beta^2)^{\frac{1}{2}} - a\right\}/2$, $Q(\xi) = \left\{(\xi^2 + ia\xi + \beta^2)^{\frac{1}{2}} + (\xi^2 + k^2)^{\frac{1}{2}}\right\}\Big/\left\{2(\xi^2 + ia\xi + \beta^2)^{\frac{1}{2}}\right\}$ and

$$Q_-(\xi) = \exp\left\{-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln Q(x)}{x - \xi}\,dx\right\}, \quad \mathrm{Im}(\xi) < 0 .$$

Combining Eqs. (5.9) and (5.7) we see that

$$\frac{\bar{f}_1(\xi)}{2} = \left\{\frac{M_-(\xi)/N_-(\xi)}{M_-(o)/N_-(o)}\right\} \frac{\bar{f}_{10}(\xi)}{2} - \frac{a^2}{\beta}\left\{i\xi\,\frac{\bar{f}_0(\xi)}{2} - e^{\pi i/4}\,a^{\frac{1}{2}}(\xi - i a_1)^{\frac{1}{2}}\,Q_-(\xi)\right\}. \tag{5.10}$$

We might recall in considering this equation that $\bar{f}_0(\xi)/2$ is the skin friction when $U_1 = 0$, and $\bar{f}_{10}(\xi)/2$ is the skin friction when $U_0 = 0$, $(\omega\nu)^{\frac{1}{2}} > U_1$.

To invert Eq. (5.10) in general is a rather difficult task; however it is possible to obtain expressions for $f_1(x)$ for small x and for large x. First let us consider $\bar{f}_{11}(\xi)/2$. It is convenient to use the expression for $\bar{f}_{11}(\xi)/2$ given in Eq. (5.6). As $\xi \longrightarrow 0$, (i.e. $x \longrightarrow \infty$) the bracketed quantity in Eq. (5.6) is invariant with changes in a, and as a matter of fact approaches the value one. Here the first approximation to $f_{11}(x)/2$ for large x is simply

$$\frac{f_{11}(x)}{2} \sim \frac{f_{10}(x)}{2} = \left(\frac{\omega}{\nu}\right)^{\frac{1}{2}} e^{\pi i/4} \qquad \text{as } x \longrightarrow \infty. \tag{5.11}$$

On the other hand $M_-(\xi)/N_-(\xi) \longrightarrow 1$ as $\xi \longrightarrow \infty$ and hence we have

$$\frac{f_{11}(x)}{2} \sim \frac{N_-(o)}{M_-(o)}\;\frac{f_{10}(x)}{2},$$

$$\sim \frac{N_-(o)}{M_-(o)}\left\{\frac{2}{\pi x}\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}} e^{\pi i/8} \qquad \text{as } x \longrightarrow 0\,. \tag{5.12}$$

The ratio $N_-(o)/M_-(o)$ is a function of the parameter $\sigma^2 = \omega\nu/(c\,U_0)^2$. It is computed in Appendix D where we obtain upon letting $N_-(o)/M_-(o) = \exp\{I(\sigma)\}$ that for $\sigma \geqslant 1/2$

$$I(\sigma) = -\frac{i}{\pi}\sum_{n=0}^{\infty}\frac{(\gamma/\sigma)^{2n+1}}{(2n+1)^2},$$

and for $\sigma \leqslant 1/2$

$$I(\sigma) = -\frac{\pi i}{8} - \frac{1}{2}\ln(2\sigma) + \frac{i}{\pi}\sum_{n=0}^{\infty}\frac{(\sigma/\gamma)^{2n+1}}{(2n+1)^2},$$

where $\gamma = 2^{-1} e^{\pi i/4}$. From these expressions for $I(\sigma)$ we can already see the effect of the mean flow on the oscillatory motion. For instance as $U_0 \longrightarrow \infty$, (i.e. $\sigma \longrightarrow 0$), $I(\sigma) \longrightarrow \left\{ -\pi i/8 - 2^{-1} \ell n\,(2\sigma) \right\}$ and upon substituting in Eq. (5.12) we see that the phase advanced in $f_{11}(x)/2$ is cancelled. We shall discuss this in more detail after we have obtained appropriate representations for $f_{12}(x)$. The real and imaginary parts of $I(\sigma)$ are given graphically in Fig. 5.

Now $\bar{f}_{12}(\xi)$ is given by Eq. (5.9). To determine $f_{12}(x)$ for small x we must know the behaviour of $\bar{f}_{12}(\xi)$ as $\xi \longrightarrow \infty$. In particular we must study $Q_-(\xi)$ for ξ large. This is done in Appendix E where we obtain the result that

$$Q_-(\xi) = 1 - \frac{G_1}{2\pi i} \; \frac{1}{\xi} + 0\left(\frac{1}{\xi^2}\right) \tag{5.13}$$

where

$$G_1 = -i a_1 \left\{ -\frac{\delta}{1-\delta}\,\pi i + 1 + \left(\frac{2}{1-\delta} - \frac{1-\delta}{2i\delta^{1/2}}\right) \log \frac{1+i\delta^{1/2}}{1-i\delta^{1/2}} \right\}, \tag{5.14}$$

and $\delta = a_1/a_2$, $a_2 = \left\{(a^2 + 4\beta^2)^{1/2} + a\right\}/2$. Upon substituting for $Q_-(\xi)$ and $f_0(\xi)$ in Eq. (5.9) we obtain that for large ξ

$$\frac{\bar{f}_{12}(\xi)}{2} = \frac{a^{1/2}\, a_1}{2\beta^2} \left\{ 1 + \frac{e^{\pi i/2}\, G'}{\pi} \right\} \frac{1}{(i\xi)^{1/2}} + 0\,(\xi^{-3/2}) \; , \tag{5.15}$$

where G' is the bracketed quantity in Eq. (5.14). Finally using the definitions of a_1, a_2, δ, β^2 and σ Eq. (5.15) can be written as

$$\frac{\bar{f}_{12}(\xi)}{2} = \frac{H(\sigma)}{2} \left(\frac{a}{i\xi}\right)^{1/2} + 0\,(\xi^{-3/2}) \, , \tag{5.16}$$

where

$$H(\sigma) = 1 + \frac{1}{\pi} \left\{ \left[2i - \frac{(1-\delta)^2}{2\,\delta^{1/2}} \right] \log \frac{1+i\delta^{1/2}}{1-i\delta^{1/2}} + i(1-\delta) \right\}$$

and $\delta = 1 + i\left\{(1 + 4i\sigma^2)^{1/2} - 1\right\}/2\sigma^2$. Consequently

$$\frac{f_{12}(x)}{2} = \frac{H(\sigma)}{2} \left(\frac{c\,U_0}{\nu \pi x}\right)^{1/2} + 0\,(x^{1/2}) \quad \text{as } x \longrightarrow 0. \tag{5.17}$$

We are now in a position to discuss the behaviour of $f_1(x)/2$ for small x. We have from Eqs. (5.12) and (5.17) that for x small

$$\frac{f_1(x)}{2} = e^{I(\sigma)} \left\{\frac{2}{\pi x}\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}} e^{\pi i/8} + \frac{H(\sigma)}{2}\left(\frac{c\,U_0}{\nu\,\pi x}\right)^{\frac{1}{2}} + 0(x^{\frac{1}{2}}),$$

$$= D(\sigma)\left(\frac{c\,U_0}{\pi\nu x}\right)^{\frac{1}{2}} + 0(x^{\frac{1}{2}})\ , \tag{5.18}$$

where

$$D(\sigma) = \frac{H(\sigma)}{2} + (2\sigma)^{\frac{1}{2}}\ e^{I(\sigma)}\ e^{\pi i/8}\ .$$

Thus near the leading edge of the flat plate the total skin friction is

$$\tau = \mu U_0 \left(\frac{c\ U_0}{\pi\nu x}\right)^{\frac{1}{2}} + \mu U_1 D(\sigma)\left(\frac{c\,U_0}{\pi\nu x}\right)^{\frac{1}{2}} e^{i\omega t}\ . \tag{5.19}$$

As $U_0 \longrightarrow \infty$, or $\omega \longrightarrow 0$ the parameter $\sigma \longrightarrow 0$ and $D(\sigma) \rightarrow \{(3/2 - 7\sigma^2/12\pi) + i4\sigma^2/\pi^2\}$ and hence when $\sigma = 0$ (i.e. $\omega = 0$)

$$\tau = \mu U_0 \left(\frac{c\,U_0}{\pi\nu x}\right)^{\frac{1}{2}} \left(1 + \frac{3}{2}\ \frac{U_1}{U_0}\right)\ . \tag{5.20}$$

This result is in agreement with Eq. (3.13). As a matter of fact it will be shown, at the end of this section, using Eq. (5.10) that the above result is valid for all x when $\sigma = 0$.

On the other hand when $\sigma \longrightarrow \infty$, i.e. $\omega \longrightarrow \infty$ we have $D(\sigma) \longrightarrow (2\sigma)^{\frac{1}{2}} \{1 + (1 - i)/(2\pi\sigma)\}\ e^{\pi i/8}$ and hence

$$\tau = \mu U_0 \left(\frac{c\,U_0}{\pi\nu x}\right)^{\frac{1}{2}} + \mu U_1 \left\{\frac{2}{\pi x}\left(\frac{\omega}{\nu}\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}} e^{i(\omega t + \pi/8)}\ . \tag{5.21}$$

This is in agreement with Eq. (4.12) as could have been anticipated. As σ increases from 0 to ∞, $\arg\{D(\sigma)\}$ and hence the phase advance in the fluctuating skin at the leading edge increases monotonically from 0 to $\pi/8$. This result is presented graphically in Fig. 7. Also the modulus of $D(\sigma)$ is given in Fig. 8. It can be seen from Fig. 8 that the modulus decreases for a short range of σ

as σ increases. This decrease is extremely slight, of the order of two or three in one hundred and fifty. However the asymptotic formula for $D(\sigma)$ for small σ bears out the numerical calculations. There is apparently no obvious physical reason why the amplitude of the skin friction should decrease slightly as σ increases from 0 to roughly 1/2.

It remains now to consider the large x behaviour of $f_{12}(x)$. (We have already shown that $f_{11}(x)/2 \longrightarrow (\omega/\nu)\, e^{\pi i/4}$ as $x \longrightarrow \infty$, see Eq. (5.11). So we must now consider the small ξ behaviour of $\bar{f}_{12}(\xi)$. This is a fairly laborious task and is perhaps not worth a detailed analysis. It can be seen from the definition of $Q_-(\xi)$ that $Q_-(o)$ is finite and hence $(\xi - ia_1)^{\frac{1}{2}}\, Q_-(\xi)$ behaves like $\xi^{\frac{1}{2}}$ for small ξ at worst. Also $i\xi\, \bar{f}_0(\xi)/2$ behaves like $(i\xi)^{\frac{1}{2}}$ so $\bar{f}_{12}(\xi)/2$, (see Eq. 5.9), will consist of terms of the type $(i\xi)^{\frac{1}{2}}$, $(i\xi)^{\frac{3}{2}}$, and so on, at worst. Hence $f_{11}(x)/2$ will be made up of terms like $x^{-3/2}$, $x^{-5/2}$, for x large. Consequently as long as $a_1 \neq 0$, (i.e. $\omega \neq 0$) these terms should die out more quickly than those contributed by $f_{11}(x)$; hence

$$f_1(x) \longrightarrow \left(\frac{\omega}{\nu}\right) e^{\pi i/4}, \quad \text{as } x \longrightarrow \infty \tag{5.22}$$

This result is to be expected on physical grounds since, as long as σ^2 is bounded away from zero, far enough down the plate the flow should consist of a shear wave superimposed on the mean flow. As σ^2 increases, (i.e. as ω increases or U_0 decreases) the shorter the distance down the plate we must go in order for (5.22) to be valid. In the limiting case $U_0 = 0$ this distance is approximately given by $(\omega/2\nu)^{\frac{1}{2}}\, x = 2.5$.

Finally we shall investigate the case $\omega = 0$. As has been pointed out earlier (see Sec. 3) we expect in this limiting case that $f_1(x) = (3/2)\,(c\,U_0/\pi\nu x)^{\frac{1}{2}}$. If we set $\omega = 0$ and hence $\beta^2 = 0$ in $M(\xi)$ (see the definition of $M(\xi)$ immediately following Eq. (5.4) we see that $M(\xi)$ becomes identical with $K(\xi)$ and hence upon examining Eqs. (5.4) and (3.9) and (3.10) we obtain

$$\frac{f_{11}(x)}{2} = \frac{f_0(x)}{2} = \left(\frac{c\,U_0}{\pi\nu x}\right)^{\frac{1}{2}} . \tag{5.23}$$

To compute $f_{12}(x)/2$ when $\beta^2 = 0$ we turn our attention to Eq. (5.9). It is simple to show by an examination of $Q(\xi)$ and $R_-(\xi)$ (defined in Appendix C) that when $\beta^2 = 0$ and hence $a_1 = 0$ that the bracketed quantity in Eq. (5.9) vanishes. Hence

$$\lim_{\beta^2 \to 0} \frac{\bar{f}_{12}(\xi)}{2} = \lim_{\beta^2 \to 0} \frac{d}{d\beta^2} \left\{ \beta^2 \frac{\bar{f}_{12}(\xi)}{2} \right\} .$$

This computation is straightforward and we obtain

$$\frac{\bar{f}_{12}(\xi)}{2} = \frac{a^{\frac{1}{2}}}{2(i\,\xi)^{\frac{1}{2}}} \tag{5.24}$$

when $\beta^2 = 0$. Consequently

$$\frac{f_{12}(x)}{2} = \frac{1}{2}\left(\frac{c\,U_0}{\pi\nu x}\right)^{\frac{1}{2}} , \tag{5.25}$$

and adding Eqs. (5.23) and (5.24) we obtained the desired result

$$\frac{f_1(x)}{2} = \frac{3}{2}\left(\frac{c\,U_0}{\pi\nu x}\right)^{\frac{1}{2}} ,$$

when $\omega = 0$.

To summarize briefly we have found that when $(\omega\nu)^{\frac{1}{2}} >> U_0$ and U_1 the mean motion and the oscillatory motion are independent. The total skin friction is given by Eqs. (4.12) and (4.8). Asymptotic expressions valid near the leading edge and far down the plate are given by Eqs. (4.13) and (4.14). It is found that the phase advance in the time dependent skin friction increases from $\pi/8$ at the leading edge to $\pi/4$ in a distance of approximately $(\omega/2\nu)^{\frac{1}{2}}\, x = 2.5$. Finally when the interaction of the mean flow and the time dependent flow can not be neglected a result valid near the leading edge and one valid far downstream are obtained for the oscillating skin friction. These results are given

in Eqs. (5.19), (5.20), (5.21) and (5.22). It is found that the phase advance in the time dependent skin friction at the leading edge increases from 0 to $\pi/8$ as the parameter $c\,U_0/(\omega\nu)^2$ increases from 0 to ∞. A graph of the phase advance versus this parameter is given in Fig. 7. Far down the plate one obtains the usual shear flow solution (see Eq. (5.22)).

Appendix A

Computation of $N_-(\xi)$

In the strip $-k < \mathrm{Im}(\xi) < k$, $N(\xi)$ is analytic and non-vanishing. Hence using Cauchy's Integral Theorem we may write

$$\ln N(\xi) = \ln\left\{N_+(\xi)\cdot N_-(\xi)\right\} = \frac{1}{2\pi i}\int_C \frac{\ln N(\xi)}{z-\xi}\,dz\ , \qquad \text{(A.1)}$$

where C is the complete contour indicated in Fig. (1), (i.e. $C = \Gamma_1 + L_1 + \Gamma_2 + L_2$). If we allow the ends of this contour to approach infinity at an equal rate, the integrals over Γ_1 and Γ_2 will converge, and the contributions from the integrations over L_1 and L_2 will vanish giving

$$\ln N_+(\xi) + \ln N_-(\xi) = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{\ln N(z)\,dz}{z-\xi} + \frac{1}{2\pi i}\int_{\Gamma_2}\frac{\ln N(z)}{z-\xi}\,dz\,. \qquad \text{(A.2)}$$

The first integral in the above equation is uniformly convergent for $\mathrm{Im}(\xi) > -k$ and hence represents an analytic function of ξ there. A similar statement is true for the second integral. Associating these integrals with $N_+(\xi)$ and $N_-(\xi)$ respectively, we have

$$\ln N_-(\xi) = \frac{1}{2\pi i}\int_{\Gamma_2}\frac{\ln\left\{(z^2+\beta^2)^{\frac{1}{2}} + (z^2+k^2)^{\frac{1}{2}}\right\}}{z-\xi}\,dz\ . \qquad \text{(A.3)}$$

If we confine ξ so that $\mathrm{Im}(\xi) < 0$ we can take Γ_2 to be the real axis in evaluating (A.3). Let

$$\frac{F(\xi)}{2\pi i} = -\frac{d\left\{\ln N_-(\xi)\right\}}{d\xi}\ ,$$

then

$$F(\xi) = \int_{-\infty}^{\infty}\frac{\ln\left\{(x^2+\beta^2)^{\frac{1}{2}} + (x^2+k^2)^{\frac{1}{2}}\right\}}{(x-\xi)^2}\,dx\ .$$

Upon integrating by parts and writing this as the sum of two integrals from $-\infty$ to 0, and 0 to ∞ and then making the usual change of variables we obtain

$$F(\xi) = \int_0^{\infty}\frac{2x^2\,dx}{(x^2-\xi^2)\,(x^2+k^2)^{\frac{1}{2}}\,(x^2+\beta^2)^{\frac{1}{2}}} \qquad \text{(A.4)}$$

In the limit as $k \longrightarrow 0$ the value of this integral is

$$F(\xi) = \frac{1}{(\xi^2 + \beta^2)^{\frac{1}{2}}} \ln \frac{\beta + (\xi^2 + \beta^2)^{\frac{1}{2}}}{\beta - (\xi^2 + \beta^2)^{\frac{1}{2}}} \quad . \tag{A.5}$$

Now we can specify $N_-(\xi)$ precisely by

$$N_-(\xi) = \exp \left\{ - \frac{1}{2\pi i} \int_0^{\xi} F(\xi)\, d\xi \right\} . \tag{A.6}$$

The lower limit in the above integral could be chosen differently but this merely multiplies $N_-(\xi)$ by a constant; the reciprocal of this constant would then multiply $N_+(\xi)$.

Appendix B

Evaluation of $f_{10}(x)/2$

Substituting in Eq. (4.4) for $N_-(\xi)$, and using the usual inversion formula we have letting $k \longrightarrow 0$, $\alpha \longrightarrow 0$ that

$$\frac{1}{2} f_{10}(x) = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{i\xi} \exp\left\{-\frac{1}{2\pi i}\int_0^{\xi} F(\xi)\, d\xi\right\} d\xi \,, \qquad \text{(B.1)}$$

where the path of integration must pass below the origin. Making the change of variables $s = \xi/\beta$, $x' = \beta x$ and noting that for $x > 0$ we can deform our path of integration to an integration over S (see Fig. 2) leads to

$$\frac{1}{2} f_{10}(x) = \frac{\beta}{2\pi} \int_S \frac{e^{isx'}}{is} G(s)\, ds, \qquad \text{(B.2)}$$

where

$$\begin{aligned} G(s) &= \exp\left\{-\frac{1}{2\pi i}\int_0^s \frac{1}{(1+s^2)^{\frac{1}{2}}} \ell n \frac{1+(1+s^2)^{\frac{1}{2}}}{1-(1+s^2)^{\frac{1}{2}}}\, ds\right\}, \\ &= \exp\left\{-\frac{1}{\pi i}\int_0^s \frac{1}{(1+s^2)^{\frac{1}{2}}} \ell n \frac{1+(1+s^2)^{\frac{1}{2}}}{e^{i\pi/2} s}\, ds\right\}. \end{aligned} \qquad \text{(B.3)}$$

As indicated in Fig. 2, the contour S is broken up into the four parts S_1, S_2, S_3, S_4, plus the enclosures about the branch points at $s = i$ and the origin. It is not difficult to show, that as the radii of these enclosures are allowed to approach zero, that there are no contributions due to the integrations around the branch point $s = i$, and that the integration around the origin contributes 2π. On the S_1 we have the following:

S_1: $s = e^{-3\pi i/2}\, y$; for $y > 1$, $(1 - y^2)^{\frac{1}{2}} = e^{-\pi i/2}\,(y^2 - 1)^{\frac{1}{2}}$

$$G(s) = -\,i\left\{y + (y^2 - 1)^{\frac{1}{2}}\right\}^{\frac{1}{2}} \exp\left\{+\,\frac{1}{2\pi i}\int_y^{\infty} \frac{1}{(y^2-1)^{\frac{1}{2}}}\,\ell n\,\frac{(y^2-1)^{\frac{1}{2}} - i}{(y^2-1)^{\frac{1}{2}} + i}\,dy\right\},$$

S_2: $s = e^{-3\pi i/2}\, y$, $y < 1$

$$G(s) = \exp\left\{-\,\frac{1}{\pi}\int_0^{y} \frac{\ell n[1 + (1 - y^2)^{\frac{1}{2}}] - \ell n\, y}{(1 - y^2)^{\frac{1}{2}}}\,dy - i \arcsin y\right\},$$

S_3: $s = e^{\pi i/2}\, y$, $y < 1$

$$G(s) = \exp\left\{-\,\frac{1}{\pi}\int_0^{y} \frac{\ell n[1 + (1 - y^2)^{\frac{1}{2}}] - \ell n\, y}{(1 - y^2)^{\frac{1}{2}}}\,dy + i \arcsin y\right\},$$

S_4: $s = e^{\pi i/2}\, y$, for $y > 1$, $(1 - y^2)^{\frac{1}{2}} = e^{\pi i/2}\,(y^2 - 1)^{\frac{1}{2}}$

$$G(s) = i\left\{y + (y^2 - 1)^{\frac{1}{2}}\right\}^{\frac{1}{2}} \exp\left\{\frac{1}{2\pi i}\int_y^{\infty} \frac{1}{(y^2-1)^{\frac{1}{2}}}\,\ell n\,\frac{(y^2-1)^{\frac{1}{2}} - i}{(y^2-1)^{\frac{1}{2}} + i}\,dy\right\}.$$

In determining these expressions for $G(s)$ it was found convenient on S_1 and S_4 to write the integral 0 to y as the sum of the integrals 0 to 1 and 1 to ∞ minus the integral y to ∞. Combining all of these results we obtain finally

$$\frac{1}{2}\, f_{10}(x) = \frac{\beta}{\pi}\left\{\pi + \int_0^1 e^{-yx'}\, H(y)\, dy + \int_1^{\infty} e^{-yx'}\, \frac{[y + (y^2 - 1)^{\frac{1}{2}}]^{\frac{1}{2}}}{y}\, P(y)\, dy\right\}. \quad \text{(B.4)}$$

Here

$$H(y) = \exp\left\{-\,\frac{1}{\pi}\int_0^{y} \frac{\ell n[1 + (1 - u^2)] - \ell n\, u}{(1 - u^2)^{\frac{1}{2}}}\,du\right\}, \quad y < 1$$

$$= \exp\left\{-\,\frac{1}{\pi}\left[t\,\ell n \cot\frac{t}{2} + \int_0^{t} \frac{\theta}{\sin\theta}\, d\theta\right]\right\}, \quad t = \arcsin y \quad ,$$

and

$$P(y) = \exp\left\{\frac{1}{2\pi i}\int_y^\infty \frac{1}{(y^2-1)^{1/2}}\ \ell n\ \frac{(y^2-1)^{1/2}-i}{(y^2-1)^{1/2}+i}\ dy\right\}\quad , y > 1$$

$$= \exp\left\{-\frac{1}{\pi}\int_0^m \frac{\theta}{\sin\theta}\ d\theta\right\}\ , \qquad m = \text{arc}\sin(1/y)\ .$$

When y ranges from 0 to 1, t goes from 0 to $\pi/2$; similarly $1 < y < \infty$ implies $\pi/2 > m > 0$.

Appendix C

Computation of $\bar{\phi}(\xi, y)$

The x Fourier transform of a function $\phi(x, y)$ which satisfies the differential equation $(\Delta - a\partial/\partial x - \beta^2)\,\Delta\phi = 0$ and the condition $\phi(x, o) = 0$ is

$$\bar{\phi}(\xi, y) = \frac{y}{|y|}\;\frac{\exp\left\{-|y|\,(\xi^2 + k^2)^{1/2}\right\} - \exp\left\{-|y|\,(\xi^2 + ia\xi + \beta^2)^{1/2}\right\}}{A(\xi)}, \qquad (C.1)$$

where $A(\xi)$ is an arbitrary function of ξ. The condition that $\partial\Psi_2/\partial y = 0$ on $y = 0$, $x > 0$ requires $\bar{\phi}_y\ (\xi, y = 0)$ be analytic in the UHP, and the condition $\partial^2\Psi_2/\partial y^2 = 0$ on $y = 0$, $x < 0$ requires $\bar{\phi}_{yy}(\xi, y = 0)$ be analytic in the LHP. These conditions may be written as

$$\bar{\phi}_y(\xi, o) = \frac{-(\xi^2 + k^2)^{1/2} + (\xi^2 + ia\xi + \beta^2)^{1/2}}{A(\xi)} = C_+(\xi) \quad , \qquad (C.2)$$

$$\bar{\phi}_{yy}(\xi, o) = \frac{(\xi^2 + k^2) - (\xi^2 + ia\xi + \beta^2)^{1/2}}{A(\xi)} = D_-(\xi) \quad , \qquad (C.3)$$

where $C_+(\xi)$ and $D_-(\xi)$ are functions analytic in the UHP and LHP respectively. The final condition on ϕ may be expressed as $\chi_{xy}(x, o) + \phi_y(x, o)$ be finite at the origin. Since the behaviour of the transform at infinity determines the behaviour of the function at the origin (see Sec. 4 and [5]), and since differentiation with respect to x implies multiplication of the transform by $i\xi$ we may express this condition as

$$\bar{\Psi}_{2y}(x, o) = -\frac{a}{\beta^2}\left\{i\xi\,\bar{\chi}_y(x, o) + \bar{\phi}_y(x, o)\right\} \sim \xi^{-(1+\epsilon)} \quad ,$$

as $\xi \longrightarrow \infty$, where $\epsilon > 0$. Using Eqs. (3.5), (3.10), and C.2) this becomes

$$\left\{\frac{e^{-3\pi i/4}\,a^{1/2}}{2\,\xi^{1/2}} + C_+(\xi)\right\} \sim \xi^{-(1+\epsilon)} \quad , \qquad (C.4)$$

as $\xi \longrightarrow \infty$. Hence $C_+(\xi) \sim \xi^{-1/2} + 0\,[\xi^{-(1+\epsilon)}]$ as $\xi \longrightarrow \infty$.

We can determine the functions $C_+(\xi)$ and $D_-(\xi)$ by an application of the Wiener-Hopf technique. First dividing (C.3) by (C.2) gives

$$
\begin{aligned}
D_-(\xi)/C_+(\xi) &= -\left\{(\xi^2 + ia\xi + \beta^2)^{\frac{1}{2}} + (\xi^2 + k^2)^{\frac{1}{2}}\right\}, \\
&= -M(\xi) \quad ,
\end{aligned}
\tag{C.5}
$$

where $M(\xi)$ is analytic in the strip $-k < \text{Im}(\xi) < k$. It is convenient to write (C.5) as

$$
D_-(\xi)/C_+(\xi) = -Q(\xi)R(\xi) \quad , \tag{C.6}
$$

where $R(\xi) = 2(\xi^2 + ia\xi + \beta^2)^{\frac{1}{2}}$ and $Q(\xi) = M(\xi)/R(\xi)$. Now we can split $Q(\xi)$ and $R(\xi)$ into the product of factors analytic in the UHP and LHP, and as a result we obtain

$$
\frac{D_-(\xi)}{Q_-(\xi)R_-(\xi)} = -C_+(\xi)Q_+(\xi)R_+(\xi) = P(\xi) \quad , \tag{C.7}
$$

where $P(\xi)$ is an entire function. The factors $R_-(\xi)$, $R_+(\xi)$, and $Q_-(\xi)$ are given by

$$
\begin{aligned}
R_-(\xi) &= (\xi - ia_1)^{\frac{1}{2}} \quad , \\
R_+(\xi) &= 2(\xi + ia_2)^{\frac{1}{2}} \quad , \\
Q_-(\xi) &= \exp\left\{-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln Q(x)}{x - \xi}\,dx\right\} \quad \text{for } \text{Im}(\xi) < 0 \, .
\end{aligned}
\tag{C.8}
$$

Here $a_1 = \left\{(a^2 + 4\beta^2)^{\frac{1}{2}} - a\right\}/2$ and $a_2 = \left\{(a^2 + 4\beta^2)^{\frac{1}{2}} + a\right\}/2$. To determine $P(\xi)$ we must first anticipate that $Q_-(\xi) = 0(1)$ as $\xi \to \infty$, and hence since $Q(\xi) \to 1$ as $\xi \to \infty$ the same is true of $Q_+(\xi)$. Also since $D_-(\xi)$ is proportional to the transform of the skin friction we expect $D_-(\xi) \sim 0(\xi^{\frac{1}{2}})$ as $\xi \to \infty$. Using these order conditions plus Eqs. (C.8) and the information about $C_+(\xi)$ we see that $P(\xi)$ must be a constant, say P'. Then

$$
C_+(\xi) = -\frac{P'}{Q_+(\xi)R_+(\xi)} \, , \quad D_-(\xi) = P'Q_-(\xi)R_-(\xi) \quad . \tag{C.9}
$$

To determine P' we substitute the above expression for $C_+(\xi)$ in Eq. (C.4). Noting that $R_+(\xi) \sim 2\xi^{-1/2}$ and $Q_-(\xi) = 1$[+] plus terms of higher order for large ξ it is clear that $P' = a^{1/2} e^{-3\pi i/4}$. Consequently

$$\bar{\phi}_{yy}(\xi, o) = e^{-3\pi i/4} a^{1/2} Q_-(\xi) R_-(\xi) \quad . \tag{C.10}$$

[+]As was pointed out earlier we anticipate that $Q_-(\xi)$ is $0(1)$ as $\xi \longrightarrow \infty$; and since $Q_-(\xi)$ is determined only up to a constant multiplier we may take the leading term to be one. Actually the expression for $Q_-(\xi)$ as given in (C.8) does lead to the desired expression when ξ is large. This will be seen in Appendix E.

Appendix D

Evaluation of $I(\sigma)$

The functions $N_-(\xi)$ and $M_-(\xi)$ are defined by

$$N_-(\xi) = \exp\left\{\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\ln\left[(z^2+\beta^2)^{1/2}+(z^2+k^2)^{1/2}\right]}{z-\xi}\,dz\right\},$$

and

$$M_-(\xi) = \exp\left\{\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\ln\left[(z^2+iaz+\beta^2)^{1/2}+(z^2+k^2)^{1/2}\right]}{z-\xi}\,dz\right\}.$$

If we restrict ξ so that $\mathrm{Im}(\xi) < 0$, then

$$\frac{N_-(\xi)}{M_-(\xi)} = \exp\left\{+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\ln\frac{(x^2+iax+\beta^2)^{1/2}+(x^2+k^2)^{1/2}}{(x^2+\beta^2)^{1/2}+(x^2+k^2)^{1/2}}\,\frac{dx}{x-\xi}\right\}. \tag{D.1}$$

In the limit as $\xi \longrightarrow 0$, our integration path must be indented to go above the origin; i.e. the path of integration is from $-\infty$ to $-\epsilon$, then a semi-circle of radius ϵ in the UHP, and then the real axis from ϵ to ∞. In the limit as $\epsilon \longrightarrow 0$ the integration over the semi-circle is zero. Letting $r = ax/|\beta|^2$, $\sigma^2 = |\beta|^2/a^2 = \omega\nu/(c\,U_0)^2$, $N_-(0)/M_-(0) = \exp I(\sigma)$, and noting that as $k \longrightarrow 0$, $(x^2+k^2)^{1/2} = |x|$ we obtain from (D.1)

$$I(\sigma) = \frac{1}{2\pi i}\int_0^{\infty}\ln\frac{\left[i(1+r)+\sigma^2 r^2\right]^{1/2}+\sigma r}{\left[i(1-r)+\sigma^2 r^2\right]^{1/2}+\sigma r}\,\frac{dr}{r}. \tag{D.2}$$

This integral can be evaluated by first computing $dI/d\sigma$ and then integrating with respect to σ. We have

$$\frac{dI}{d\sigma} = \frac{1}{2\pi i}\int_0^{\infty}\left\{\frac{1}{[\sigma^2 r^2+ir+i]^{1/2}} - \frac{1}{[\sigma^2 r^2-ir-i]^{1/2}}\right\}dr,$$

$$= -\frac{1}{2\pi i\sigma}\ln\frac{\sigma+\gamma}{\sigma-\gamma}, \tag{D.3}$$

where $\gamma = e^{\pi i/4}/2$. Consequently

$$I(\sigma) = \frac{1}{2\pi i}\int_{\sigma}^{\infty} \ln \frac{r+\gamma}{r-\gamma}\frac{dr}{r} \quad . \tag{D.4}$$

The constant of integration (i.e. the upper limit in the above integral) is determined by noting that when $a = 0$, $N_-(o) = M_-(o)$ hence $I(\sigma) = 0$, but $a = 0$ corresponds to $\sigma = \infty$.

For $\sigma \geq 1/2$, $I(\sigma)$ can be evaluated directly by expanding the integrand in a power series in γ/σ and then integrating termwise. We obtain

$$I(\sigma) = -\frac{i}{\pi}\sum_{n=0}^{\infty}\frac{(\gamma/\sigma)^{2n+1}}{(2n+1)^2}\,, \qquad \sigma \geq 1/2 \quad . \tag{D.5}$$

For $\sigma < 1/2$ the evaluation of (D.4) is slightly more difficult. It is convenient to write (D.4) as an integral from γ to ∞ plus an integral σ to γ, then we obtain

$$I(\sigma) = -\frac{1}{2}\ln(2\sigma) + \frac{1}{2\pi i}\int_{\sigma}^{\gamma}\frac{1}{r}\ln\frac{\gamma+r}{\gamma-r}\,dr \quad . \tag{D.6}$$

After writing the integral in (D.6) as one from 0 to γ minus an integral from 0 to σ we obtain

$$\begin{aligned} I(\sigma) &= -\frac{\pi i}{8} - \frac{1}{2}\ln(2\sigma) - \frac{1}{2\pi i}\int_0^{\sigma}\frac{1}{r}\ln\frac{\gamma+r}{\gamma-r}\,dr \\ &= -\frac{\pi i}{8} - \frac{1}{2}\ln(2\sigma) + \frac{i}{\pi}\sum_{n=0}^{\infty}\frac{(\sigma/\gamma)^{2n+1}}{(2n+1)^2}\,, \qquad \sigma \leq 1/2 \quad . \end{aligned} \tag{D.7}$$

Appendix E

Asymptotic Expansion of $Q_-(\xi)$

The function $Q_-(\xi)$ as defined by (C.8) is

$$Q_-(\xi) = \exp\left\{- \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln Q(x)}{x - \xi}\, dx\right\} \qquad \text{for } \mathrm{Im}(\xi) < 0\,, \qquad (E.1)$$

where

$$Q(\xi) = \frac{(\xi^2 + ia\xi + \beta^2)^{\frac{1}{2}} + (\xi^2 + k^2)^{\frac{1}{2}}}{2(\xi^2 + ia\xi + \beta^2)^{\frac{1}{2}}} \quad .$$

Except for branch points at $\xi = ik$ and $\xi = ia_1$, $Q(\xi)$ has no other singularities in the UHP. Hence using Cauchy's Integral Theorem the integral in (E.1) can be converted to an integral over T (see Fig. 6). So if we write $Q_-(\xi) = \exp\{-G(\xi)/2\pi i\}$ we have

$$G(\xi) = - \int_T \frac{\ln Q(z)}{z - \xi}\, dz \quad . \qquad (E.2)$$

For large ξ, $G(\xi)$ has asymptotic representation of the form

$$G(\xi) = G_0 + G_1/\xi + \ldots \qquad (E.3)$$

where

$$G_0 = \lim_{\xi \to \infty} G(\xi) \quad , \qquad G_1 = - \lim_{\xi \to \infty} \xi^2 \frac{dG}{d\xi} \quad . \qquad (E.4)$$

It is not difficult to show from (E.2) that G_0 is zero. Performing the limiting operation indicated in (E.4) we see that

$$G_1 = - \int_T \ln Q(z)\, dz \quad . \qquad (E.5)$$

It is convenient to write $Q(z)$ as

$$Q(z) = \frac{1}{2}\left\{1 + V(z)\right\}\;; \quad V(z) = \frac{(z + ik)^{\frac{1}{2}}\,(z - ik)^{\frac{1}{2}}}{(z - ia_1)^{\frac{1}{2}}\,(z + ia_2)^{\frac{1}{2}}} \quad .$$

To evaluate (E.5) in the limit as $k \longrightarrow 0$, we make the change of variables $x = z/ia_1$, and let $\delta = a_1/a_2$. Then our branch cut lies on the real axis from 0 to 1, and our path of integration is clockwise. The function $V(z)$ has the values

$$V(x) = \frac{x\, e^{-\pi i/2}}{(1-x)\quad (x + 1/\delta)^{1/2}}\,, \quad V(x) = \frac{x\, e^{+\pi i/2}}{(1-x)^{1/2}(x+1/\delta)^{1/2}}\,,$$

on the upper and lower sides of the branch cut. We obtain

$$G_1 = -\, i a_1 \int_0^1 \ell n\, \frac{1 + i q(x)}{1 - i q(x)}\, dx; \quad q(x) = \frac{x}{(1-x)^{1/2}(x + 1/\delta)^{1/2}} \quad . \tag{E.6}$$

The integral in Eq. (E.6) can be evaluated by integration by parts and some straightforward algebraic manipulation. We obtain

$$G_1 = -\, i a_1 \left\{ -\, \frac{\delta}{1-\delta}\, \pi i + 1 + \left(\frac{2}{1-\delta} - \frac{1-\delta}{2i\delta^{1/2}} \right) \ell n\, \frac{1 + i\,\delta^{1/2}}{1 - i\,\delta^{1/2}} \right\} .$$

So for ξ large

$$Q_-(\xi) = 1 - \frac{G_1}{2\pi i}\;\frac{1}{\xi} + 0\left(\frac{1}{\xi^2}\right) ,$$

where G_1 is defined above.

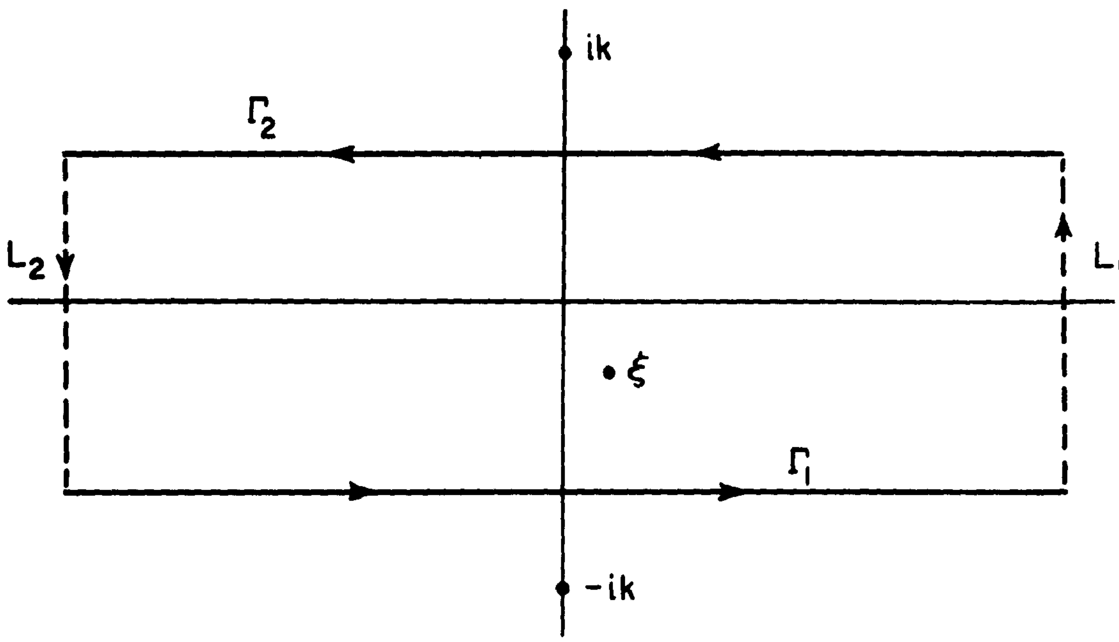


FIGURE 1

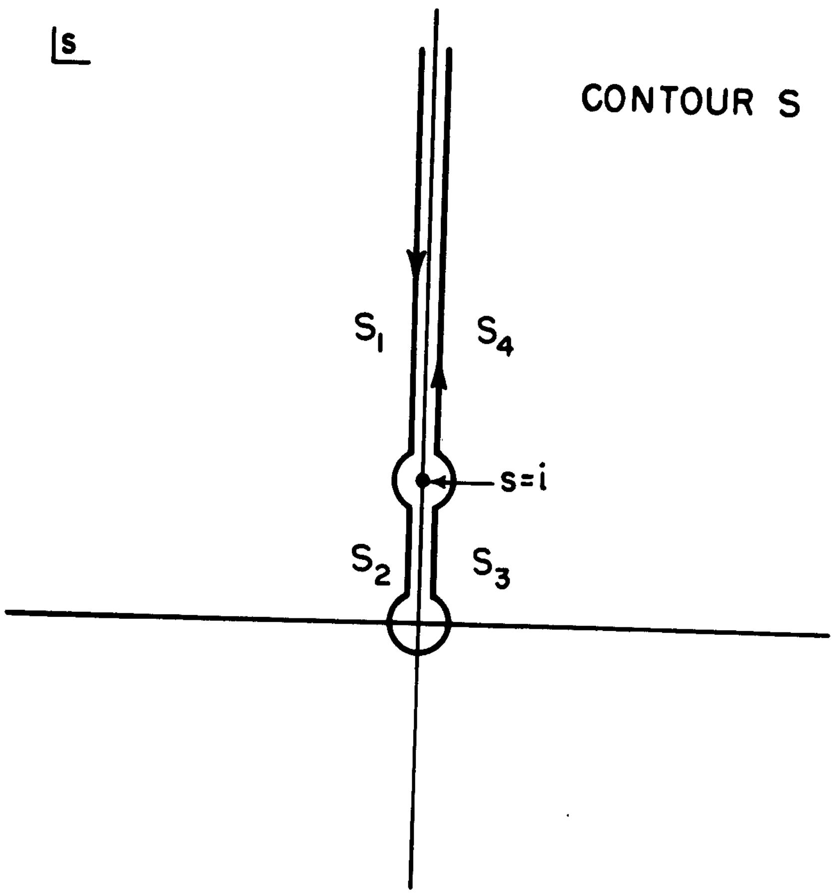


FIGURE 2

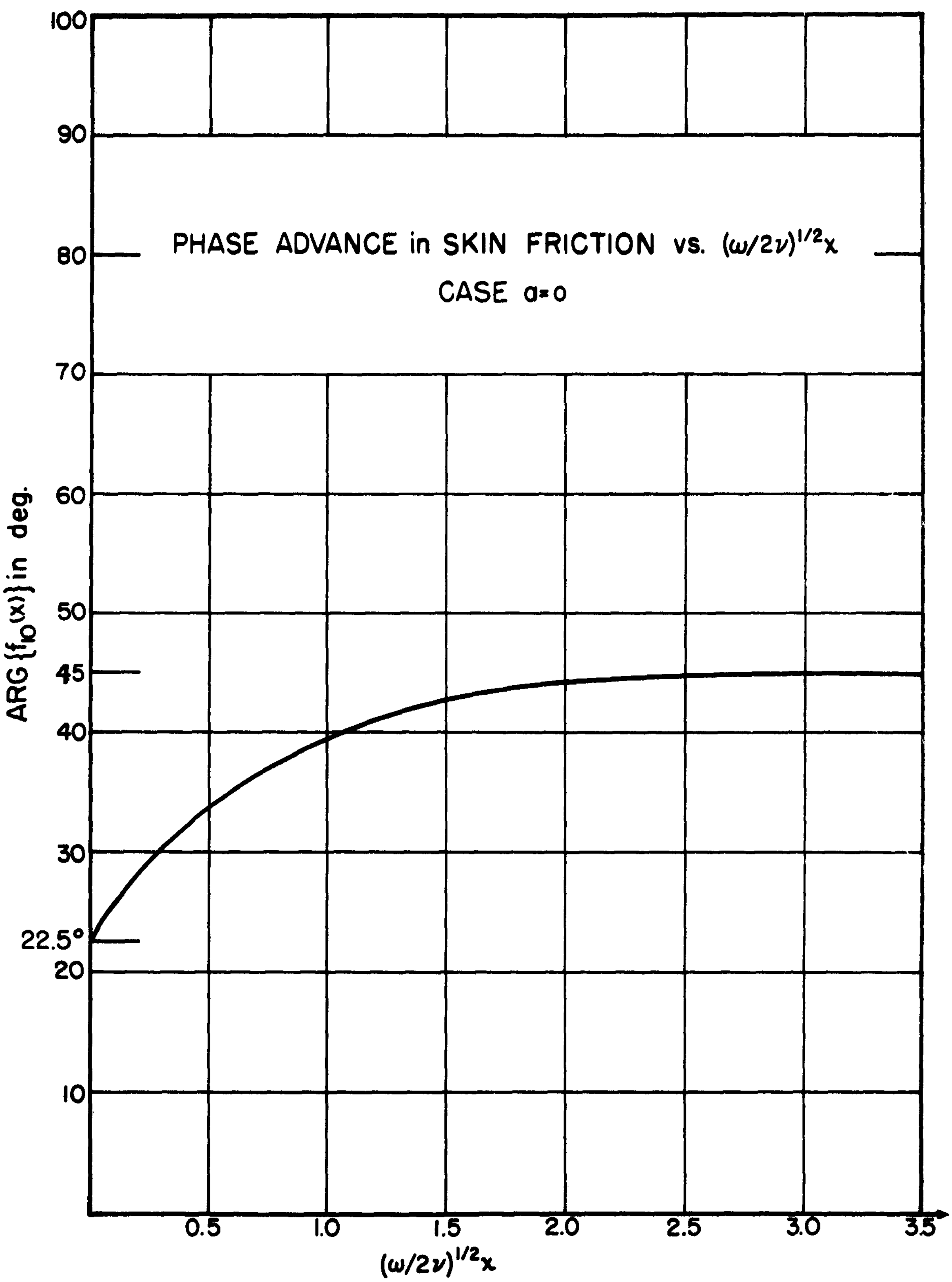

PHASE ADVANCE in SKIN FRICTION vs. $(\omega/2\nu)^{1/2}x$
CASE a=0
ARG$\{f_{10}(x)\}$ in deg.
100
90
80
70
60
50
45
40
30
22.5°
20
10
0.5
1.0
1.5
2.0
2.5
3.0
3.5
$(\omega/2\nu)^{1/2}x$

FIGURE 3

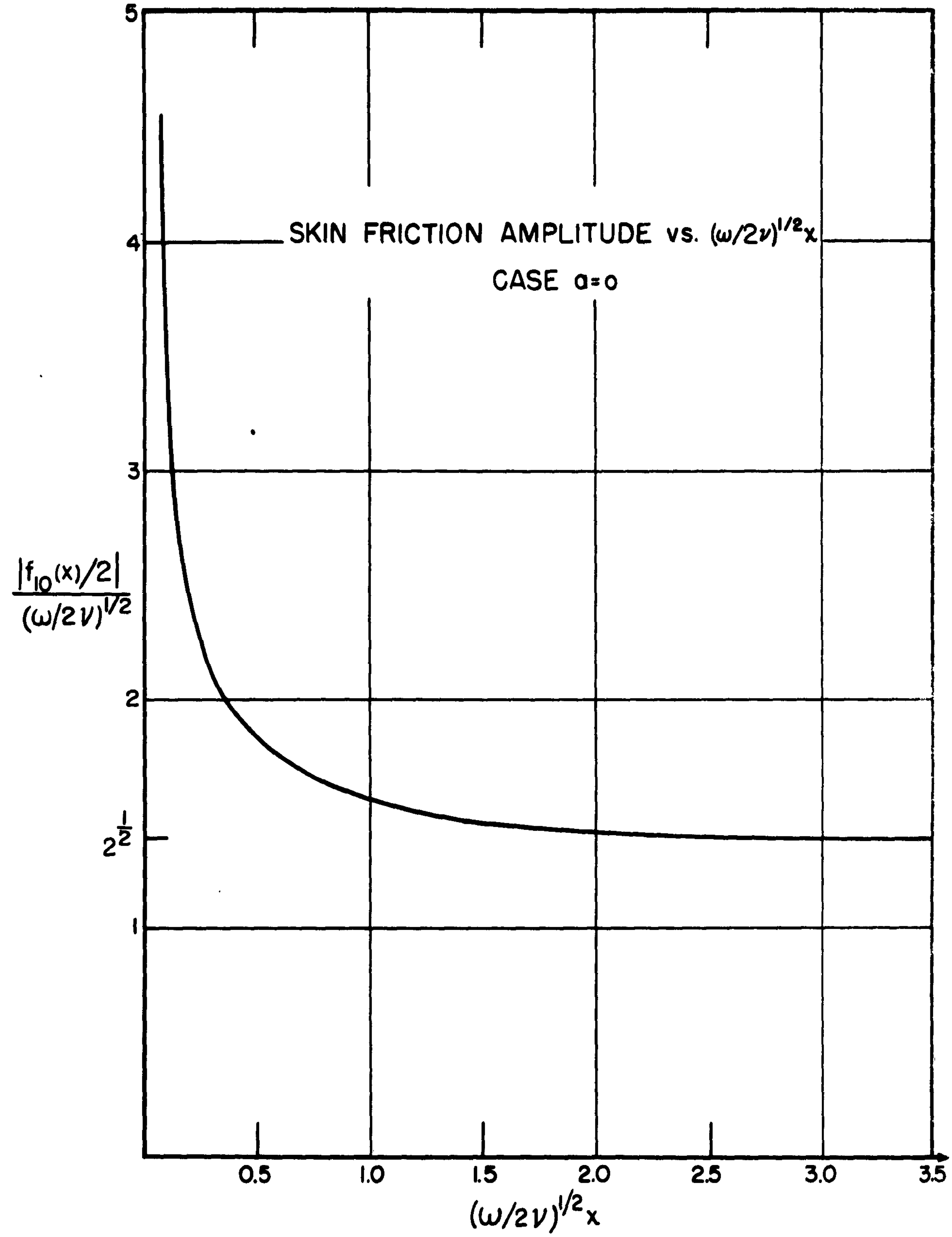
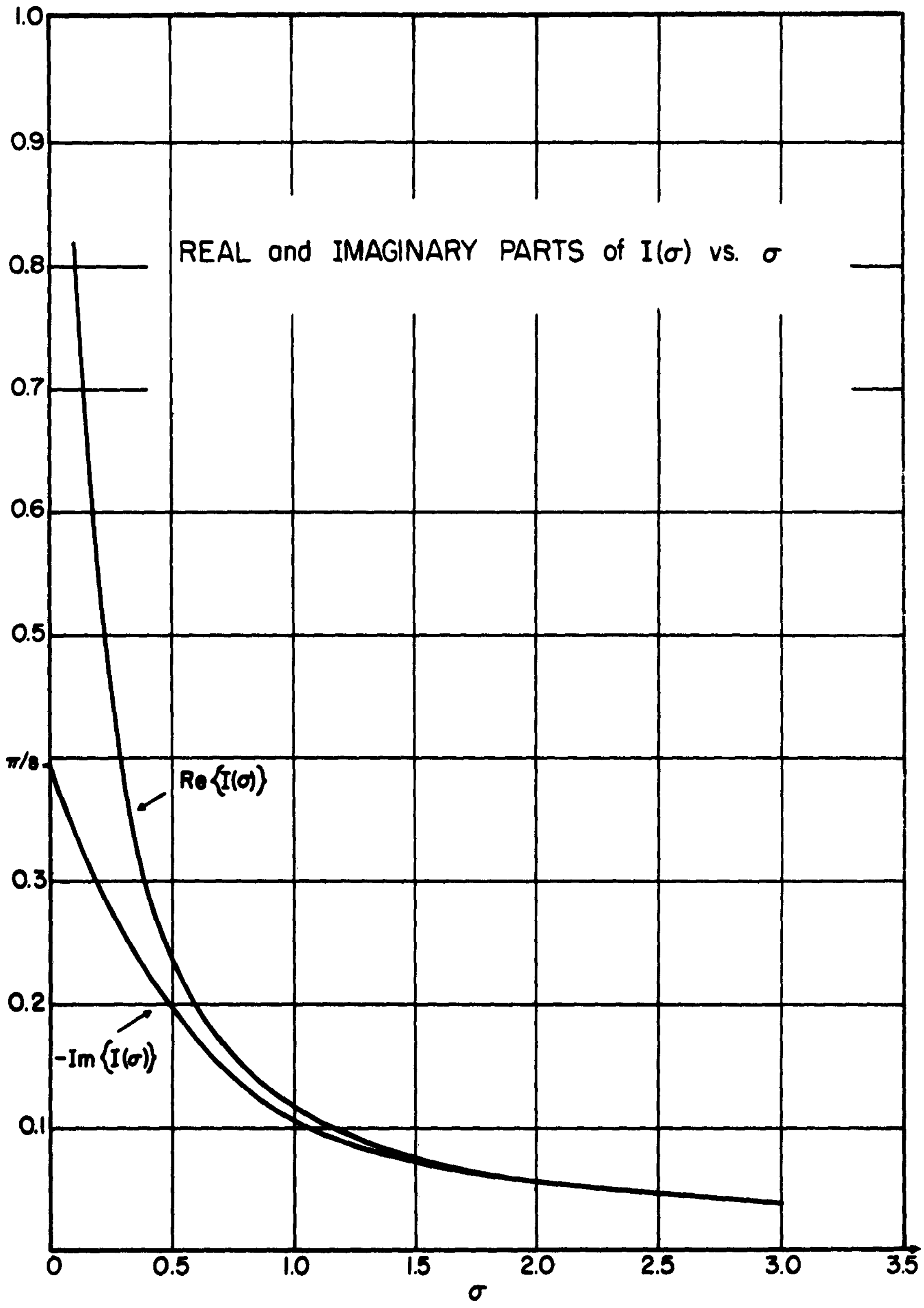

SKIN FRICTION AMPLITUDE vs. $(\omega/2\nu)^{1/2}x$
CASE a=0
5
4
3
2
$2^{\frac{1}{2}}$
1
$\frac{|f_{10}(x)/2|}{(\omega/2\nu)^{1/2}}$
0.5
1.0
1.5
2.0
2.5
3.0
3.5
$(\omega/2\nu)^{1/2}x$

FIGURE 4



REAL and IMAGINARY PARTS of I(σ) vs. σ
1.0
0.9
0.8
0.7
0.6
0.5
π/8
0.3
0.2
0.1
Re{I(σ)}
-Im{I(σ)}
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
σ

FIGURE 5

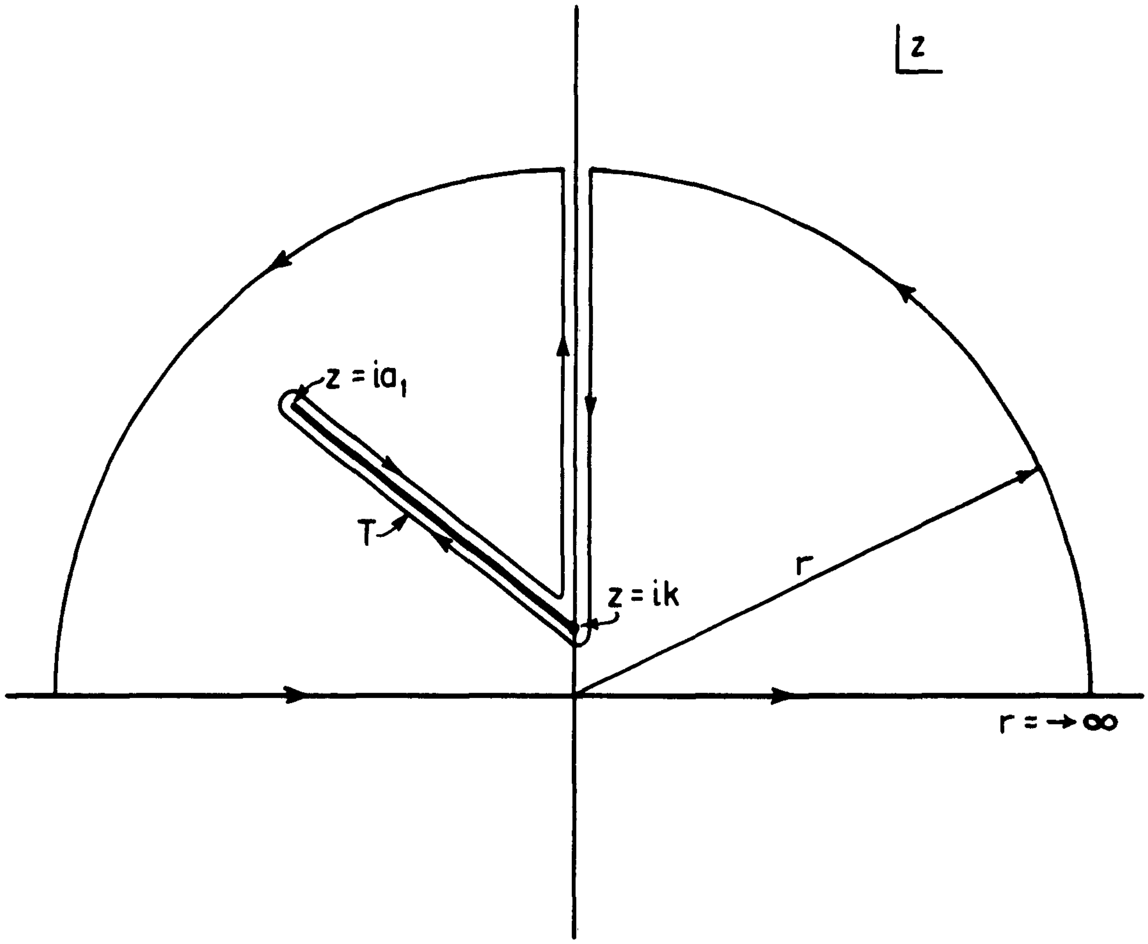


FIGURE 6

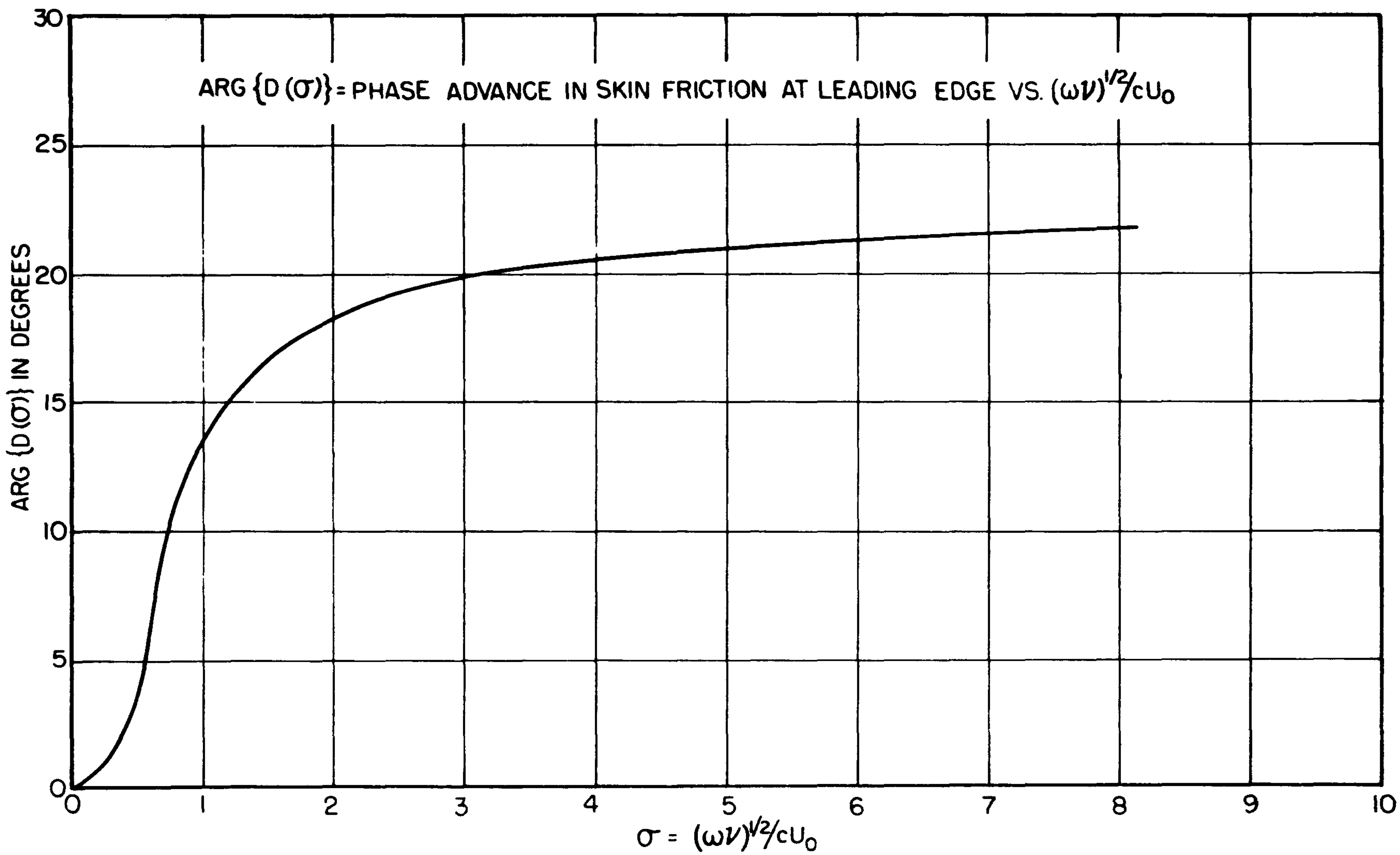

ARG {D (σ)} = PHASE ADVANCE IN SKIN FRICTION AT LEADING EDGE VS. (ων)^{1/2}/cU0
ARG {D (σ)} IN DEGREES
30
25
20
15
10
5
0
0
1
2
3
4
5
6
7
8
9
10
σ = (ων)^{1/2}/cU0

FIGURE 7

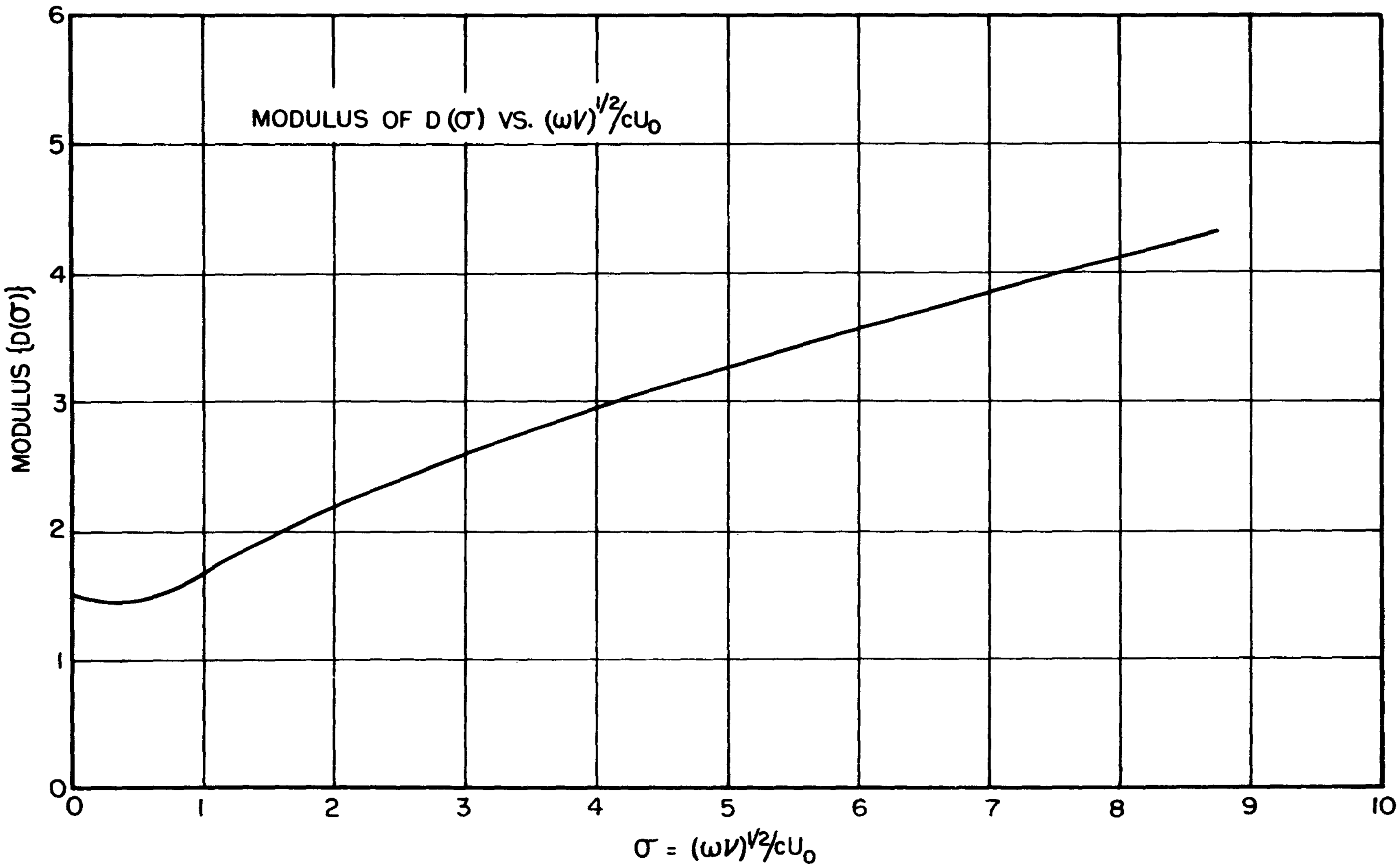


FIGURE 8

References:

1. J. A. Lewis and G. F. Carrier, Some remarks on the flat plate boundary layer, Quart. App. Math., 7, pp. 228-233, (1949).

2. G. F. Carrier, On slow viscous flow, Report to the O.N.R. NR-062-163, March 1953.

3. G. F. Carrier, The mechanics of the Rijke tube, Quart. App. Math., 12, pp. 383-395, (1955).

4. P. Lagestrom, J. Cole, and L. Trilling, Problems in the theory of viscous compressible fluids, GALCIT Report ONR Contract N6 ONR - Task Order VIII, March 1949.

5. G. Doetsch, Theorie und anwendung der Laplace-transformation, Dover Publications, New York, N. Y., pp. 243 - 246 (1943).

6. H. Lamb, Hydrodynamics, 6th edition, Dover Publications, New York, N. Y., p. 620, (1945).

7. M. J. Lighthill, The response of laminar skin friction and heat transfer to fluctuations in the stream velocity, Proc. Roy. Soc. of London, A, 224 pp. 1 - 23, (1954).

8. F. K. Moore, Unsteady laminar boundary layer flow, NACA TN 2471 September, 1951.

9. S. Ostrach, Compressible laminar boundary layer and heat transfer for unsteady motions of a flat plate, NACA TN 3569, November, 1955.

10. J. T. Stuart, A solution of the Navier-Stokes and energy equations illustrating the response of skin friction and temperature of an infinite plate thermometer to fluctuations in the stream velocity, Proc. Roy. Soc. of London, A, 231, pp. 116 - 130, (1955).

11. Lord Rayleigh, Theory of Sound, 2, end edition, Dover Publications, New York, N. Y., pp. 224 - 234, (1945).